Topological Dynamics of Transcendental Entire Functions

Thesis submitted in accordance with the requirements of the

University of Liverpool for the degree of Doctor in Philosophy by

Helena Mihaljević-Brandt

September 3, 2009

Page ii is blank in the original.

Abstract

Helena Mihaljević-Brandt

In this dissertation we discuss various current problems in holomorphic dynamics. We give a transcendental analogue to Douady's theorem which states that if f is a polynomial of degree ≥ 2 with connected Julia set $\mathcal{J}(f)$ then every periodic point in $\mathcal{J}(f)$ is the landing point of at least one and at most finitely many dynamic rays, each of which is periodic. We prove exactly the same conclusion for a large class of transcendental entire maps, including all geometrically finite maps with finite order of growth.

Furthermore, we show that under more restrictive conditions, this result can be considerably strengthened: if f is a strongly subhyperbolic transcendental entire map, then there is a hyperbolic map $g : z \mapsto f(\lambda z)$ with connected Fatou set, and a semiconjugacy Φ between g and f on their Julia sets. Moreover, Φ restricts to a conjugacy between the escaping sets of g and f. This result enables us to describe the Julia set of any strongly subhyperbolic map f as a quotient of the Julia set of a particularly simple hyperbolic map in the same parameter space. We obtain two interesting corollaries: The escaping set of every strongly subhyperbolic map is disconnected, answering in particular a question by W. Bergweiler. Furthermore, it follows for a large class of strongly subhyperbolic maps, including all such maps of finite order, that their Julia set is a pinched Cantor bouquet, consisting of dynamic rays and their endpoints.

In the last part of the thesis, we consider nonescaping-hyperbolic functions. We prove that if a holomorphic family of entire maps is approximated in a dynamically sensible way by a sequence of holomorphic families of entire functions, then the nonescaping-hyperbolic components in the respective parameter spaces converge as kernels. Similar results are known for more restrictive (explicit) cases and we embed the underlying ideas in a much more general and natural context.

Acknowledgements

First and foremost, I want to thank my supervisor, Lasse Rempe, for being a great teacher, for his friendship and constant help and encouragement. Special thanks go to Mary Rees, not only for her professional, but in particular for her thoughtful personal support. I also want to thank Adam Epstein for many inspiring and helpful conversations. Thanks go to my colleagues at the University of Liverpool, in particular to my friend Nathan, who was there whenever I needed his help. I am grateful for financial support from EPSRC and the Department of Mathematics, in particular for the possibility of attending many interesting workshops and conferences. Many thanks go to Núria Fagella, Antonio Garijo, Xavier Jarque, Jörn Peter and Jordi Taixés — the Barcelona dynamics group — for their support and many instructive discussions during my term in Barcelona, especially to Núria and Xavi for being such caring hosts. I would also like to thank Walter Bergweiler, Arnaud Chéritat, Kealey Dias, Freddie Exall, Toby Hall, Sara Munday, Radhakrishnan Nair, Carsten Petersen, Phil Rippon, Dierk Schleicher, Gwyneth Stallard and Saadet Yurttas for many thoughtprovoking discussions and helpful suggestions and advices.

Doing a PhD in Liverpool, at a department with great working atmosphere, and getting the chance to see and to live in different parts of the world, was a very valuable experience that I am grateful for. At the same time, it was very challenging to be away from family and friends. So my special thanks go to my parents Milica and Branko, my sister Kristina, my niece Karla, and particularly to my partner Abel for their endless love and support.

Contents

1 Introduction

This thesis lies in the area of (1-dimensional) holomorphic dynamics — the study of the behaviour of iterated holomorphic maps on Riemann surfaces — including rational maps on the Riemann sphere $\widehat{\mathbb{C}}$ as well as entire functions on $\mathbb{C}$. The focus of this work will be transcendental entire maps. The objects of main dynamical interest are the *Fatou set* $\mathcal{F}(f)$ of the map f, defined as the maximal open set where the iterates $\{f^n\}_{n\in\mathbb{N}}$ form a normal family in the sense of Montel, and its complement the *Julia set* $\mathcal{J}(f)$.

The first fundamental results in iteration theory of holomorphic functions were achieved in the early 20th century. The mathematicians Pierre Fatou and Gaston Julia are considered to be the "principal fathers" of holomorphic dynamics, where especially Fatou's memoires [Fat19] are universally regarded as the fount of the modern theory. In the beginnings, the main focus was the global study of iterated rational maps, often in connection with functional equations. Nevertheless, Fatou was able to transfer some of his results to transcendental entire functions. Some of his observations led to intense research interest that is still active to this day. For instance, in the Julia sets of certain explicit functions, he discovered curves consisting of points which tend to infinity under iteration; it was proved just recently that this phenomenon occurs in a very general setting [RRRS]. Following a period lasting several decades where there was little research output, holomorphic dynamics received much attention in the 1980's. Various groundbreaking results, such as Sullivan's theorem on the absence of wandering domains in rational dynamics, and Douady and Hubbard's work on the Mandelbrot set, energized the development of the field. The invention of computer algorithms that made it possible to visualize highly complicated dynamical behaviour gave the progress an additional drive.

At that time, the main focus was still centred on the study of rational dynamics. The description of the *escaping set*

$$I(f) := \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\}$$

of a map f has played an important role, in particular in polynomial dynamics: the escaping set $I(f)$ of a polynomial f (of degree $d \geq 2$) constitutes an open subset of the Fatou set, while the Julia set equals the boundary of $I(f)$. Furthermore, when $\mathcal{J}(f)$ is connected, which is the case if and only if no (finite) singular value escapes to ∞, the so-called *Böttcher isomorphism* Φ_f maps $I(f)$ onto the exterior of the closed unit disk $\overline{\mathbb{D}}$, conjugating f and $z \mapsto z^d$. The curves that arise as preimages of straight rays from $\partial\mathbb{D}$ to ∞ under Φ_f are called *dynamic rays*; they provide a natural foliation of the escaping set of such a polynomial, thus the *landing behaviour* of dynamic rays, i.e., their limiting behaviour within $\mathbb{C}$, is strongly connected to the structure of the Julia set.

The invention of dynamic rays as a tool in complex dynamics goes back to Douady and Hubbard [Dou93]. One can say that since then, dynamic rays, and their landing properties in particular, have been an essential ingredient in the success of polynomial dynamics. One of the fundamental results in this area, which goes back to Douady, states that each repelling or parabolic periodic point of a polynomial with connected Julia set is the landing point of at least one and at most finitely many periodic dynamic rays [Mil06, Theorem 18.11]. It was one of the main goals of this thesis to establish an analogy to this theorem for transcendental entire maps.

For a transcendental entire function f, ∞ is an essential singularity. The effect of this is that the escaping set of f is no longer open, hence there is no conformal isomorphism near ∞ — like the Böttcher isomorphism — that would induce a foliation of $I(f)$ with curves to ∞. Nevertheless, it has long been known that for certain classes of transcendental entire functions there exist curves in the escaping set which can be seen as analogs of dynamic rays [DGH86, DT86]. These curves are often referred to as *hairs*, but here we will call them *dynamic rays* of f to stress the analogy to the polynomial case.

Eremenko was the first to perform a systematical exploration of the escaping set of (non-explicit) transcendental entire functions [Ere89]. Among other results, he proved that the escaping set $I(f)$ of such a function is never empty and that — as for polynomials — $\mathcal{J}(f) = \partial I(f)$. He also showed that in many cases, including all maps that are of interest in this thesis, the stronger relation $\mathcal{J}(f) = \overline{I(f)}$ holds, which is in stark contrast to the behaviour of polynomials. Furthermore, he asked whether curves in the escaping set, like those that had already been observed by Fatou, occur in general, that is, whether the escaping set of an *arbitrary* transcendental entire function f has the property that every point in $I(f)$ can be connected to ∞ by a curve in $I(f)$. Recently Rottenfußer, Rückert, Rempe and Schleicher [RRRS] proved that for a large class of maps, containing every finite order transcendental entire function whose set of singular values is bounded, the escaping set consists of dynamic rays and (some of) their endpoints. This provides us with a large class of functions where we can study the topology of Julia sets by looking at landing properties of dynamic rays. This approach to the study of Julia sets and escaping sets has been used with great success in certain families of transcendental entire maps such as the exponential family $E_\lambda(z) = \lambda e^z$ or the cosine family $F_{a,b}(z) = ae^z + be^{-z}$.

The underlying intention of this dissertation is to broaden the knowledge of topological dynamics, first obtained for certain explicit families of transcendental entire maps, to more general classes of functions. We start with the following natural question suggested by Douady's theorem for polynomials and the results of [RRRS]. Suppose that f is a finite order entire function whose singular set $S(f)$ is bounded, and suppose also that $S(f) \cap I(f) = \emptyset$. Is every repelling or parabolic periodic point of f the landing point of a periodic ray of f? Even for exponential and cosine maps, this question is still open — a partial result on exponential maps can be found in [Rem06]. However, Schleicher and Zimmer [SZ03] obtained a positive answer for exponential maps satisfying certain dynamical assumptions. In Chapter 4, we generalize this statement, under similar conditions, to a much larger class of transcendental entire functions. We call a map f *geometrically finite* if $S(f) \cap \mathcal{F}(f)$ is compact and if the intersection of the postsingular set $P(f)$ and $\mathcal{J}(f)$ is finite.

Theorem 1.1 (Periodic points are landing points). *Let f be geometrically finite and assume that f has finite order. Then, for any repelling or parabolic periodic point z of f, there is a periodic dynamic ray landing at z.*

In fact, the conditions that f is geometrically finite and has finite order can be weakened, as we will see in Chapter 4.

Theorem 1.1 implies in particular that each singular value in the Julia set of a map f to which the theorem applies is the landing point of some periodic dynamic ray. Using these rays, we define a dynamically natural partition of the Julia set, as done for exponential and cosine maps in [SZ03, Sch07a], which is useful for studying the topological dynamics of f in combinatorial terms. As our main application, we prove in Chapter 5 the remaining part of the analogy to Douady's theorem.

Theorem 1.2. *Let f be a geometrically finite map for which every periodic point in $\mathcal{J}(f)$ is the landing point of a periodic ray. Then every periodic point* *in $\mathcal{J}(f)$ is the landing point of at most finitely many (periodic) dynamic rays.*

Under more restrictive function-theoretic conditions, Theorem 1.1 can be considerably strengthened. Indeed, if $f = F_{a,b} : z \mapsto a\,e^z + b\,e^{-z}$ is a cosine map that is postcritically strictly preperiodic, i.e., for which both critical values are strictly preperiodic, then Schleicher showed in [Sch07a] that *every* point $z \in \mathbb{C}$ is either on a dynamic ray or the landing point of a dynamic ray. In Chapter 6 we generalize this result to any *strongly subhyperbolic* function f of finite order. (A function is said to be subhyperbolic if it is geometrically finite and has no parabolic cycles; f is called strongly subhyperbolic if it is subhyperbolic, $\mathcal{J}(f)$ contains no asymptotic values and the degree of f on $\mathcal{J}(f)$ is uniformly bounded.) More precisely, the following result which can be considered as the main achievement in the present work (with many interesting implications) describes the Julia set of *any* strongly subhyperbolic entire function as a quotient of the Julia set of a particularly simple *nonescaping-hyperbolic* function in the same parameter space. (An entire map f is called nonescaping-hyperbolic if $P(f)$ is a compact subset of $\mathcal{F}(f)$.)

Theorem 1.3. *Let f be a strongly subhyperbolic transcendental entire map, and let $\lambda \in \mathbb{C}$ be such that $g(z) := f(\lambda z)$ is nonescaping-hyperbolic with connected Fatou set. Then there exists a continuous surjection $\phi : \mathcal{J}(g) \to \mathcal{J}(f)$, such that*

$$f(\phi(z)) = \phi(g(z))$$

for all $z \in \mathcal{J}(g)$. Moreover, ϕ restricts to a homeomorphism between the escaping sets $I(g)$ and $I(f)$.

The hypothesis will be automatically satisfied whenever λ is sufficiently small. Also, any two maps g and g' as in the theorem are quasiconformally conjugate on their Julia sets [Rem], so it will be sufficient to prove the theorem for any such map.

A nonescaping-hyperbolic transcendental entire map with connected Fatou set is said to be of *disjoint type*. For simple families, such as $z \mapsto \lambda \sinh z$, the dynamics of functions of disjoint type is well-understood. Hence Theorem 1.3 extends this understanding to all strongly subhyperbolic functions in these families. This has the following particularly interesting consequence: While it is well-known that the Julia set $\mathcal{J}(f)$ of a transcendental entire function f can be the whole complex plane, as far as we know, there is no function with this property for which the topological dynamics has been completely understood. Theorem 1.3 provides such a description for all such maps that are strongly subhyperbolic, including functions such as $z \mapsto \pi \sinh z$. For this particular map, we will elaborate a detailed description of the topological dynamics in Chapter 6.5.

It is known that two distinct exponential maps with (strictly) preperiodic asymptotic values are not conjugate on their escaping sets [Rem06]. This rigidity is caused by the interference of the asymptotic value with the topology of the escaping set. Rempe asked whether two postcritically strictly preperiodic cosine maps can be conjugate on their escaping sets [Rem06, Question 12.1]. Theorem 1.3 and the mentioned result in [Rem] on disjoint type maps give an affirmative answer to this question.

Furthermore, we give an answer to the question of Bergweiler (personal communication) whether the escaping set of a postcritically strictly preperiodic cosine map is connected: this is not the case. More generally, it is known that the escaping set of a disjoint type map is always disconnected, so we obtain the following corollary, settling Bergweiler's question for all strongly subhyperbolic maps.

Corollary 1.4. *The escaping set of a strongly subhyperbolic transcendental entire function is disconnected.*

Moreover, it is known that the Julia set of any map of disjoint type and finite order is a *Cantor bouquet* [Bar07], i.e., it is homeomorphic to a *straight brush* in the sense of [AO93].

Corollary 1.5. *Let f be a strongly subhyperbolic map of finite order. Then $\mathcal{J}(f)$ is a* pinched Cantor bouquet*; that is, the quotient of a Cantor Bouquet by a closed equivalence relation defined on its endpoints.*

In fact, similar to Theorem 1.1, the assumption of finite order can be weakened using results from [RRRS]. For instance, Corollary 1.5 is true for every strongly subhyperbolic map that can be written as a finite composition of finite order maps with bounded singular sets.

Every pinched Cantor bouquet as in Corollary 1.5 consists of dynamic rays and their endpoints [RRRS]. Hence this implies the following result on landing behaviour of dynamic rays.

Corollary 1.6. *Let f be as in Corollary 1.5. Then every dynamic ray of f lands and every point in $\mathcal{J}(f)$ is either on a dynamic ray or the landing point of a dynamic ray.*

As aforementioned, when f is supposed to be a cosine map $F_{a,b}(z) = a\,\mathrm{e}^z + b\,\mathrm{e}^{-z}$ with strictly preperiodic critical values, Corollary 1.6 has already been shown by Schleicher [Sch07a]. Nonetheless, his results do not explain the topological embedding of the escaping set of such a map in the complex plane.
For nonescaping-hyperbolic maps, Theorem 1.3 is due to Rempe, and our proof is in the spirit of the ideas presented in [Rem]. However, the attempt to transfer the construction in the nonescaping-hyperbolic case to the setting of strongly subhyperbolic maps fails due to the existence of singular values in Julia sets. This obstruction is overcome by studying Julia sets as subsets of *hyperbolic Riemann orbifolds.* These can be thought of as images of $\mathbb{D}$ under branched coverings, for which the set of critical values is discrete and "tame". This yields a description of a Riemann orbifold as a Riemann surface together with a discrete set of *ramified points*, each of which has finite ramification value. The use of orbifolds in dynamics goes back to Thurston and has been used with great success by Douady and Hubbard in their work on subhyperbolic rational maps. The following result is not only crucial for the proof of Theorem 1.3 but also interesting in its own right, since it provides us with a global estimate of the hyperbolic metric on certain hyperbolic Riemann orbifolds.

Theorem 1.7. *Let $K > 1$ and let z_i be an infinite sequence of points satisfying $|z_j| < |z_{j+1}| \leq K|z_j|$. Let $\mathcal{O}$ be the complex plane $\mathbb{C}$ whose ramified points are the points z_i with ramification value 2.*
Then the density $\rho_{\mathcal{O}}(z)$ of the hyperbolic metric on $\mathcal{O}$ satisfies

$$\frac{1}{\rho_{\mathcal{O}}(z)} \leq O(|z|) \quad \text{as} \quad z \to \infty.$$

The adoption of successful concepts from rational or polynomial iteration into the transcendental case — as done in the proof of Theorem 1.3 — is a general strategy in holomorphic dynamics. The roots of this approach seem to lie in the approximation of transcendental entire maps by polynomials. Perhaps the most prominent such example is the approximation of the exponential family $E_\lambda(z) = \lambda\, e^z$ by the polynomials $P_{n,\lambda}(z) = \lambda\,(1 + z/n)^n$, as first investigated by Devaney, Goldberg and Hubbard [DGH86]. The result was a nice connection between the respective parameter spaces: the authors proved pointwise convergence of nonescaping-hyperbolic components (i.e., connected components of the set of those $\lambda \in \mathbb{C}$ for which E_λ or $P_{d,\lambda}$, respectively, is nonescaping-hyperbolic) as well as convergence of certain external rays to curves called "hairs" in exponential parameter space (see also [BDH$^+$00]). One could say that this point of view has provided an important conceptional basis for much subsequent work on the exponential family.

For such strong connections to hold, it is crucial that — as in the above example — the sets of singular values do not have too much freedom during the iteration process. In Chapter 7, we embed the underlying approximation idea into a general setup. We introduce a metric χ_{dyn} on the space $\text{Hol}_b^*(\mathbb{C})$ of all nonconstant, nonlinear entire maps, which is dynamically more sensible in the sense that two maps are close in this metric if their locally uniform distance *and* the Hausdorff distance between their sets of singular values is small. We will say that (families of) maps converge *dynamically* if they converge with respect to χ_{dyn}. Now, let M be a complex manifold. For every $n \in \mathbb{N} \cup \{\infty\}$ let $\mathscr{F}_n = \{f_{n,\lambda}\} \subset \text{Hol}_b^*(\mathbb{C})$ be a family of entire functions that depend holomorphically on $\lambda \in M$. Assume that for every n, the singular values of all maps in $\mathscr{F}_n$ form bounded sets and are holomorphically parametrized by M, and that $\mathscr{F}_n \to \mathscr{F}_\infty$ dynamically. Our main result on this topic is the following.

Theorem 1.8. *If $\widetilde{H}$ is a kernel of a sequence of nonescaping-hyperbolic components of $\mathscr{F}_n$, then exactly one of the following statements holds:*

(*i*) *The map $f_\lambda \in \mathscr{F}_\infty$ is not nonescaping-hyperbolic for any $\lambda \in \widetilde{H}$.*

(*ii*) *There is a nonescaping-hyperbolic component H_∞ of $\mathscr{F}_\infty$ such that $\widetilde{H} = H_\infty$.*

A *kernel* of a sequence of open connected subsets of M is defined in a similar fashion as for domains in $\mathbb{C}$, i.e., in the sense of Carathéodory.

Theorem 1.8 is a natural generalization of work by Krauskopf and Kriete who studied nonescaping-hyperbolic maps with finitely many singular values [KK97]. The first case in Theorem 1.8 does indeed occur. Nevertheless, parameters that belong to a kernel define maps which exhibit certain stability. Under the same assumptions and notations as in Theorem 1.8 we prove the following result.

Theorem 1.9. *Let λ belong to a kernel $\widetilde{H}$. Then $f_\lambda \in \mathscr{F}_\infty$ is a J-stable function.*

Given an entire function f together with a sequence of entire maps f_n for which $\chi_{\text{dyn}}(f_n, f) \to 0$ when $n \to \infty$, there is a natural way — using quasiconformal equivalence classes — to construct suitable holomorphic families $\mathscr{F}_n \ni f_n$ that satisfy the assumptions of Theorem 1.8. For a lot of explicit functions, including many examples that have been of particular dynamical interest in the last few decades, non-trivial approximations in the sense of χ_{dyn} are known.

Structure of the thesis

Chapter 2 contains introductory and background material that is fundamental for the understanding of the thesis. Most concepts and results stated in this part are well-known; a proof is given for those statements for which a reference could not be located. Since the theorems presented in this dissertation address various types of holomorphic maps, we found it crucial to present in a separate section those classes of maps that are of main interest to us; this is the contents of Chapter 3. The organization of the subsequent chapters reflects the order in which the results are presented in the introduction: the goal of Chapter 4 is to show that, for a large class of maps, every periodic point in the Julia set is the landing point of a periodic dynamic ray. Chapter 5 addresses the remaining part of the analogy to Douady's polynomial theorem, namely the proof that at most finitely many rays land at every such point. Section 6 is devoted to the description of the dynamics of strongly subhyperbolic maps using functions of disjoint type. Finally, in Section 7, we consider holomorphic families of entire maps and study the behaviour of nonescaping-hyperbolic components in parameter spaces during dynamically sensible approximation. Since each of the main chapters can be considered independently, we will discuss possible open questions and problems at the end of each section rather than at the end of the thesis.

The fact that the presented results address different classes of maps is certainly an additional difficulty for the reader. Although our main focus clearly centres on transcendental entire maps, polynomials will play a role as well, and some of our constructions will be purely local. Furthermore, some of the terms we introduce for transcendental entire maps already exist for rational functions but without having the same implications, which is why we will sometimes reflect on the rational case as well. However, we will state at the beginning of every section the type of maps that is subsequently studied. We will also frequently remind the reader of the assumed setting.

2 Preliminaries

2.1 Notations

The complex plane, the Riemann sphere and the punctured plane are denoted by $\mathbb{C}$, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$, respectively. We write $\mathbb{D}$ for the unit disk and $S^1 := \partial\mathbb{D}$ for the unit circle; $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ is the punctured disk. We denote the upper half-plane by $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. A Euclidean disk centred at $c \in \mathbb{C}$ with radius r is denoted by $D_r(c)$. For two sets $A, B \subset \mathbb{C}$ we use the notation $\operatorname{dist}(A, B)$ for the Euclidean distance between A and B. If $A \subset \mathbb{C}$ then we denote the boundary and the closure of A relative to the complex plane by ∂A and $\overline{A}$, respectively. Sometimes, we will require these operators relative to the sphere $\widehat{\mathbb{C}}$; in such a case, we will write $\widehat{\partial} A$ for the boundary and $\widehat{A}$ for the closure of A. If B is an open subset of the complex plane, we will write $A \Subset B$ if the set A is compactly contained in B, i.e., if $\overline{A}$ is a compact set contained in B. For a set $A \subset \mathbb{C}$ the *fill-in* of A is defined as the union of A and the bounded components of $\mathbb{C} \setminus A$. If M is a metric space then we denote the ε-neighbourhood of a set $A \subset M$ by $U_\varepsilon(A)$. The *derived set* of a set $A \subset \mathbb{C}$ is the set of all finite limit points of A and will be denoted by A'. If $A \subset \mathbb{C}$ is an open set and $\gamma : [t_0, \infty) \to \mathbb{C}$ is a curve with $\lim_{t\to\infty} \gamma(t) = \infty$, then we say that γ is *eventually contained* in A if some *tail* $\gamma|_{[t,\infty)}$, $t \geq t_0$, of γ is contained in A. For a finite sequence (n_i) of natural numbers we write $\operatorname{lcm}\{n_i\}$ for their least common multiple.

We denote by $\operatorname{Hol}(\mathbb{C})$ the space of all maps that are entire, i.e., holomorphic everywhere in $\mathbb{C}$. Every such map is either a polynomial or a transcendental entire function. In this dissertation, we will mainly consider the space $\operatorname{Hol}^*(\mathbb{C})$ of all nonconstant, nonlinear functions $f \in \operatorname{Hol}(\mathbb{C})$. We will be particularly interested in transcendental entire maps; polynomials will only be considered in Chapter 7. However, the basic definitions will be given for all entire maps.

2.2 Background on hyperbolic geometry

A domain (i.e., an open connected set) $U \subset \mathbb{C}$ is called *hyperbolic* if $\mathbb{C} \setminus U$ contains at least two points. The *hyperbolic metric* on U is the unique complete conformal Riemannian metric on U of constant curvature -1. We denote the density of the hyperbolic metric on U by $\rho_U(z)$. To each rectifiable curve $\gamma : (a,b) \to U$ we assign the *hyperbolic length* $\ell_U(\gamma) := \int_\gamma \rho_U(z)|dz|$ of γ. For any two points $z, w \in U$ the *hyperbolic distance* $d_U(z,w)$ is the smallest hyperbolic length of a curve connecting z and w in U.

Let $f : V \to U$ be a holomorphic function between two hyperbolic domains. We denote the derivative of f with respect to the hyperbolic metric of U (where defined) by

$$\|Df(z)\|_U := |f'(z)| \cdot \frac{\rho_U(f(z))}{\rho_U(z)}.$$

Note that this expression is only meaningful when $V \cap U \neq \emptyset$. We say that f is a *contraction* for the respective hyperbolic metrics if $\rho_V(z) \geq \rho_U(f(z)) \cdot |f'(z)|$ for every $z \in V$. If equality holds then we say that f is a *local isometry.* If $\rho_V(z) > \rho_U(f(z)) \cdot |f'(z)|$ for all $z \in V$ then we say that f is a *strict contraction.* Note that a local isometry does not necessarily preserve distances; more precisely, if $f : V \to U$ is a local isometry then $d_V(z,w) = d_U(f(z), f(w))$ whenever z and w are sufficiently close but in general, only the inequality $d_V(z,w) \geq d_U(f(z), f(w))$ holds.

Theorem 2.1 (Pick's theorem, [Mil06, Theorem 2.11]). *A holomorphic map $f : V \to U$ between two hyperbolic domains does not increase the respective hyperbolic metrics. In fact, it is a local isometry if and only if f is a covering map; otherwise f is a strict contraction.*

Pick's theorem shows in particular that if $V \subsetneq U$, then $\rho_V(z) > \rho_U(z)$ for all $z \in V$. In this case, the derivative of a covering map $f : V \to U$ satisfies $\|Df(z)\|_U > 1$.

We will use the following standard estimates on the density of the hyperbolic metric of a domain $U \subset \mathbb{C}$ [Mil06, Corollary A.8]:

$$\frac{1}{2 \cdot \mathrm{dist}(z, \partial U)} \leq \rho_U(z) \leq \frac{2}{\mathrm{dist}(z, \partial U)}, \tag{2.1}$$

where the inequality on the left-hand side holds if U is simply-connected.

By the Uniformization Theorem [Hub06, Theorem 1.1.1], any hyperbolic domain U is conformally isomorphic to a quotient of the form $\mathbb{D}/\Gamma$, where Γ is a Fuchsian group acting on $\mathbb{D}$.

Let $\pi : \mathbb{D} \to U$ be a universal covering map. A *geodesic* on U is the image of a geodesic on $\mathbb{D}$. (A curve $\gamma \subset \mathbb{D}$ is a geodesic on $\mathbb{D}$ if and only if γ is an arc of a circle orthogonal to S^1.) If g is a geodesic connecting two points $z, w \in U$, then g has minimal hyperbolic length among curves connecting z and w in the same homotopy class (understood relative $\widehat{\partial} U$).

Suppose that w is an isolated point of $\widehat{\partial} U$; such a point is called a *puncture* of U. By [Hub06, Proposition 3.8.9], there exists a covering map $p : \mathbb{D}^* \to U$ such that p extends to a continuous map from $\mathbb{D}$ to $\mathbb{C}$ which sends 0 to w, and such that, for sufficiently small $\varepsilon > 0$, the restriction $p : \overline{\mathbb{D}_\varepsilon(0)} \setminus \{0\} \to U$ is one-to-one. If ε has this property, then the simple closed curve $h_\varepsilon(w) := p(\partial \mathbb{D}_\varepsilon(0))$ is called a *horocycle at* w; the component $H_\varepsilon(w)$ of $U \setminus h_\varepsilon(w)$ whose boundary is $h_\varepsilon(w) \cup \{w\}$ is called a *horosphere at* w.

It will often be important to know that we can replace any curve by a geodesic in the same homotopy class. The following statement is well-known.

Proposition 2.2. *Let $U \subset \widehat{\mathbb{C}}$ be a finitely-connected hyperbolic domain and suppose that $\gamma : [0, \infty] \to \widehat{U}$ is a curve with $\gamma\big((0, \infty)\big) \subset U$. Then there exists a unique geodesic g of U that is homotopic to γ.*

Sketch of proof. Let γ be such a curve with endpoints p and q, and let $\tilde{\gamma}$ be a lift of γ to its universal cover $\mathbb{D}$. By [EM88, Lemma 2.1], $\tilde{\gamma}$ has well-defined endpoints, say $\tilde{p}$ and $\tilde{q}$. (In the case when p and q belong to U or are punctures, this follows directly.) Replace $\tilde{\gamma}$ by the unique geodesic $\tilde{g}$ with the same endpoints. Let us now consider the projection g of $\tilde{g}$ to U. We can assume that U has a nontrivial boundary component; otherwise, U would have only punctures, and since every geodesic in $\mathbb{D}$ enters every horocycle at the endpoint in $\partial\mathbb{D}$, every such geodesic projects to a curve ending in the puncture. By a standard trick, we can map U conformally to a bounded domain. The statement then follows from Lindelöf's theorem [CL04, Theorem 2.3]. □

The following result states that geodesics on hyperbolic domains will stay away from the punctures.

Lemma 2.3 ([Jor82, Lemma 2]). *Let U be a hyperbolic domain and let w be a puncture of U. There exists $\varepsilon > 0$ such that each simple geodesic entering the horosphere $H_\varepsilon(w)$ ends at the point w.*

2.3 Background on function theory and dynamics

Throughout this paragraph, we will assume that $f \in \mathrm{Hol}^*(\mathbb{C})$. Denote by f^n the n-th iterate of $f : \mathbb{C} \to \mathbb{C}$. We say that a point $z \in \mathbb{C}$ is a *periodic point* of f if there exists an integer $n \geq 1$ such that $f^n(z) = z$. The smallest n with this property is called the *period* of z. A periodic point of period one is called a *fixed point*. We call a point z *preperiodic* under f if some image $f^n(z)$, $n \geq 1$, of z is periodic. Note that every periodic point is also preperiodic. To avoid confusion, we say that a point z is *strictly preperiodic* if it is preperiodic but not periodic. Let z be a periodic point of f of period n. The set $O^+(z) := \{z, f(z), \ldots, f^{n-1}(z)\}$ of all forward images of z is called the *(forward) orbit* or *cycle* of z, or simply a *periodic orbit* or *periodic cycle*. (More generally, we denote the forward orbit of any set $C \subset \mathbb{C}$ under f by $O^+(C) := \bigcup_{n\geq 0} f^n(C)$.) We call $\mu(z) := (f^n)'(z)$ the *multiplier* of the periodic point z. Since $\mu(f^i(z)) = \mu(z)$ for all i, we can assign to every periodic cycle a unique multiplier.

A periodic point z is called *attracting* if $0 \leq |\mu(z)| < 1$, *indifferent* if $|\mu(z)| = 1$ and *repelling* if $|\mu(z)| > 1$. An attracting periodic point z is called *superattracting* if $\mu(z) = 0$. We denote the union of all attracting periodic points of f by $\mathrm{Attr}(f)$. Since the multiplier of an indifferent periodic point is of the from $\mathrm{e}^{2\pi i t}$ with $0 \leq t < 1$, we can distinguish between *rationally* and *irrationally indifferent* points, according to whether t is rational or not. A rationally indifferent periodic point is also called *parabolic*. We denote the union of all parabolic cycles of f by $\mathrm{Par}(f)$.

If Z is an attracting periodic cycle of f of period n, then the *basin of attraction* or *attracting basin of* Z is the open set $A(Z)$ consisting of all points $z \in \mathbb{C}$ for which the successive iterates $f^n(z), f^{2n}(z), \ldots$ converge to some point of Z. If $w \in Z$ then the component $A^*(w)$ of $A(Z)$ that contains w is called the *immediate basin of attraction of* w. (The immediate basin $A^*(Z)$ of the cycle Z is then the union of the immediate basins of the points in Z.) Analogously, we define *parabolic (immediate) basins*. We denote the set of all points that converge to an attracting cycle of f by $\mathcal{A}(f)$; $\mathcal{P}(f)$ denotes the set of all points that converge nontrivially to a parabolic cycle of f.

The *local degree* $\deg(f, z_0)$ of f at the point $z_0 \in \mathbb{C}$ is the unique integer $n = n(z_0) \geq 1$ such that

$$f(z) = f(z_0) + a_n(z - z_0)^n + \mathrm{O}((z - z_0)^{n+1})$$

for $z \to z_0$, where $a_n \neq 0$. A *critical value* of f is the image of a point c with $\deg(f, c) > 1$; such a point c is called a *critical point*. The set of all critical values of f is denoted by $C(f)$. A (finite) *asymptotic value* of f is a point $a \in \mathbb{C}$ for which there exists a curve $\gamma : (0, \infty) \to \mathbb{C}$ with $\gamma(t) \to \infty$ as $t \to \infty$ such that $\lim_{t\to\infty} f(\gamma(t)) = a$. We will write $A(f)$ for the set of all asymptotic values of f. If f is a polynomial then one can easily find a constant r_f such that for any $r > r_f$, the set $\{z \in \mathbb{C} : |z| > r\}$ is mapped into itself. Hence a polynomial has no asymptotic values. Furthermore, if f is transcendental entire and w is a (Picard) exceptional value of f, i.e., the equation $f(z) = w$ has at most finitely many solutions z, then $w \in A(f)$ [GO70, Chapter 5, Theorem 1.1]. The set of *singular values* $S(f)$ of f is the smallest closed subset of $\mathbb{C}$ such that $f : \mathbb{C}\backslash f^{-1}(S(f)) \to \mathbb{C}\backslash S(f)$ is a covering map. It is obvious from the definitions that $C(f) \cup A(f) \subset S(f)$. Conversely, using lifting arguments, one can show that $S(f) = \overline{C(f) \cup A(f)}$ (see e.g. [Eps97, Section 3, p.15] or [GK86, Lemma 1.1]). Furthermore, if f, g are entire functions, then $S(g \circ f) = S(g|_{f(\mathbb{C})}) \cup \overline{g(S(f))}$. A point $z \notin S(f)$ is called a *regular value* of f. We will see in Theorem 2.11 that the dynamics of a map f strongly depends on the behaviour of its singular values.

We denote by $\mathrm{Hol}^*_b(\mathbb{C})$ the set of all those maps in $\mathrm{Hol}^*(\mathbb{C})$ whose singular sets are bounded. The *Eremenko-Lyubich class* is defined to be

$$\mathscr{B} := \{f \in \mathrm{Hol}^*_b(\mathbb{C}) : f \text{ is transcendental entire}\}.$$

The *postsingular set* of f is defined by $P(f) = \overline{\bigcup_{n\geq 0} f^n(S(f))}$.

Lemma 2.4. *Let f be a transcendental entire function. Then $P(f)$ contains at least two points.*

Proof. First observe that if $S(f) = \emptyset$ then $f : \mathbb{C} \to \mathbb{C}$ is a universal covering map and the associated group of deck transformations is trivial. But then f is a conformal automorphism of $\mathbb{C}$ and not a transcendental (entire) map, as assumed.

Now let us assume that f has only one singular value, at w say, since otherwise there is nothing to prove. For simplicity, assume that $w = 0$. A simple covering space argument shows that f is of the form $f(z) = \exp(az + b)$ for some $a \in \mathbb{C}^*, b \in \mathbb{C}$ (see e.g. [Rem03, Theorem 2.3.5]). In this case the asymptotic value

$w = 0$ is also an omitted value, so $P(f)$ contains at least two points. □

The order of a map $f \in \mathrm{Hol}^*(\mathbb{C})$ is defined to be

$$\rho(f) := \limsup_{r\to\infty} \frac{\log\log M(f,r)}{\log r},$$

where $M(f,r)$ denotes the maximum absolute value of $f(z)$ when $|z| = r$. Hence f has finite order if there exist constants $\rho, C > 0$ such that for all $r > 0$, $\sup_{|z|=r} |f(z)| \leq C \cdot \mathrm{e}^{r^\rho}$. To give some examples, the order of a polynomial is zero, the order of the exponential or cosine map is one and the order of the map $z \mapsto \mathrm{e}^{\mathrm{e}^z}$ is infinite. By the Denjoy-Carleman-Ahlfors Theorem [Nev53, XI §4, 269], a map of order $\rho < \infty$ has at most 2ρ asymptotic values. Let us define

$$R^3S := \{f = f_1 \circ \cdots \circ f_n : f_i \in \mathcal{B} \text{ and } \rho(f_i) < \infty \text{ for all } i\}.$$

This class of maps has been extensively studied in [RRRS]. Note that $R^3S \subset \mathcal{B}$ while a map $f \in R^3S$ does not necessarily have finite order, as the example $f(z) = \mathrm{e}^{\mathrm{e}^z}$ shows. Also note that $g \circ f \in \mathcal{B}$ is possible without $f, g \in \mathcal{B}$, as the example $f(z) = \mathrm{e}^z - z$, $g(z) = \mathrm{e}^z$ shows.

2.4 Classical results from holomorphic dynamics

Let $f \in \mathrm{Hol}^*(\mathbb{C})$. The objects of main dynamical interest related to f are the *Fatou set* $\mathcal{F}(f)$ of f, defined as the set of all points that have a neighbourhood in which $(f^n)_{n\in\mathbb{N}}$ forms a normal family in the sense of Montel, and its complement the *Julia set* $\mathcal{J}(f) := \mathbb{C} \setminus \mathcal{F}(f)$ of f.
In this paragraph, we will summarize some important theorems from complex dynamics that will be used frequently throughout the thesis.
The following characterization of Julia sets was proved for rational maps independently by Fatou and Julia; the proof for transcendental entire maps was first given by Baker [Bak68]. A more elementary proof (for rational and entire maps) has been given by Schwick [Sch97].

Theorem 2.5. *The Julia set of a function $f \in \mathrm{Hol}^*(\mathbb{C})$ equals the closure of the set of repelling periodic points of f.*

Before we state some basic properties of Fatou and Julia sets, recall that a set A is said to be *completely invariant* under a map f if $w \in A$ implies that $f(w) \in A$

and that $z \in A$ whenever $f(z) \in A$. Also, a set is said to be *perfect* if it is non-empty, closed and contains no isolated points.

Theorem 2.6 (Properties of Fatou and Julia sets, [Ber93]). *Let* $f \in \text{Hol}^*(\mathbb{C})$.

(i) $\mathcal{F}(f)$ *is open.*

(ii) $\mathcal{J}(f)$ *is perfect. Furthermore, either* $\mathcal{J}(f) = \mathbb{C}$ *or* $\mathcal{J}(f)$ *has no interior.*

(iii) $\mathcal{F}(f) = \mathcal{F}(f^n)$ *and* $\mathcal{J}(f) = \mathcal{J}(f^n)$ *for all* $n \geq 2$.

(iv) $\mathcal{J}(f)$ *and* $\mathcal{F}(f)$ *are completely invariant under* f.

Another dynamically relevant set is the *escaping set* of f, defined by

$$I(f) := \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\}.$$

A point $z \in I(f)$ is called an *escaping point.* Note that $I(f^n) = I(f)$ for all $n \geq 1$.

In a punctured neighbourhood of ∞, the behaviour of polynomials on the one hand, and transcendental entire maps on the other, is extremely different. For a transcendental entire map, ∞ is an essential singularity, hence such a map does not extend to a holomorphic (or even continuous) map on $\widehat{\mathbb{C}}$. On the other hand, a polynomial p extends to a holomorphic map $\widehat{p} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\widehat{p}(\infty) = \infty$. As is the rule for Riemann surfaces in general, we can choose a local uniformizing chart, e.g. $\psi : z \mapsto 1/z$, and compute the local degree of $\widehat{p}$ at ∞ in terms of the chart; it turns out that ∞ is a critical point (recall that p is not constant or linear). Hence we can consider ∞ as a superattracting fixed point of $\widehat{p}$. This implies that $I(p)$ is a subset of $\mathcal{F}(p)$. (In particular, $I(p)$ is open.) In fact, if p is a polynomial then $I(p)$ is a completely invariant Fatou component and $\mathcal{J}(p) = \partial I(p)$ [Mil06, Lemma 9.4].

The study of the escaping set of a transcendental entire map is much more delicate. Eremenko undertook the first systematic study of escaping sets of transcendental entire maps and proved the following relations.

Theorem 2.7 (Properties of escaping sets, [Ere89]). *The following statements hold for every transcendental entire map* f.

(i) $I(f) \neq \emptyset$.

(*ii*) $\mathcal{J}(f) = \partial I(f)$.

(*iii*) $I(f) \cap \mathcal{J}(f) \neq \emptyset$.

As aforementioned, the first two statements also hold for polynomials, while (*iii*) does not. Note that Theorem 2.7 (*iii*) implies that the escaping set of a transcendental entire map is never open, since otherwise, the iterates would form a normal family in a neighbourhood of any escaping point, implying that the escaping set must be a subset of the Fatou set.
Under the more restrictive assumption of class $\mathcal{B}$, Eremenko and Lyubich proved the following [EL92].

Theorem 2.8. *Let $f \in \mathcal{B}$ and $z \in \mathcal{F}(f)$. Then $f^n(z) \nrightarrow \infty$ when $n \to \infty$.*

Together with Theorem 2.7 (*iii*) we immediately obtain the following.

Corollary 2.9. *If $f \in \mathcal{B}$ then $\mathcal{J}(f) = \overline{I(f)}$.*

Let U be a (connected) component of $\mathcal{F}(f)$. By Theorem 2.6 (*iv*), $f^n(U)$ is contained in a component of $\mathcal{F}(f)$, which we denote by U_n. We say that U is *periodic* if there exists $n \geq 1$ such that $U_n = U$. The smallest n with this property is called the *period* of U. If $n = 1$, then we say that U is a *fixed* component of $\mathcal{F}(f)$. We call U *preperiodic* if for some $n > 1$, the domain U_n is periodic. A component which is preperiodic but not periodic is called *strictly preperiodic.* If U is not preperiodic, then we call U a *wandering domain.*
The dynamical behaviour of an entire map on the periodic components of its Fatou set is well-understood. We classify the periodic components of entire functions with bounded sets of singular values since these are of main interest to us.

Theorem 2.10 (Classification of periodic Fatou-components). *Let $f \in \text{Hol}_b^*(\mathbb{C})$ and let $U \neq I(f)$ be a periodic component of $\mathcal{F}(f)$ of period p. Then one of the following possibilities holds:*

(*i*) *U contains an attracting periodic point z_0 of period p. Then $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$. U is called the* immediate attracting basin *of z_0.*

(*ii*) *∂U contains a periodic point z_0 of period p and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$. Then $(f^p)'(z_0) = 1$ and U is called a* Leau domain *or an* immediate parabolic basin.

(*iii*) *There is an analytic homeomorphism* $\phi : U \to \mathbb{D}$ *such that* $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha} z$ *for some* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. *In this case,* U *is called a* Siegel disk.

In particular, every such component U *is simply-connected.*

Recall from Corollary 2.9 that $U = I(f)$ can occur only if f is a polynomial. For more details on Theorem 2.10 see [Ber93, Theorem 6], [EL92, Proposition 3, Theorem 1] and [Mil06, Lemma 9.4].

By Sullivan's theorem [Mil06, Theorem 16.4], rational maps have no wandering domains. The same is known for entire maps with finitely many singular values [EL92, Theorem 3]. However, it is an open question as to whether this is true for all maps in the class $\mathcal{B}$.

There is a close relationship between Fatou components and (post)singular values. The next theorem summarizes those results that are relevant to us.

Theorem 2.11 (Fatou components and singular values). *Let* $f \in \mathrm{Hol}^*(\mathbb{C})$ *and let* $C = \{U_0, \dots, U_{p-1}\}$ *be a periodic cycle of components of* $\mathcal{F}(f)$. *If* f *is a polynomial then assume that* $U_0 \neq I(f)$.

- *If* C *is a cycle of immediate attracting or parabolic basins, then* $S(f) \cap U_j \neq \emptyset$ *for some* $j \in \{0, \dots, p-1\}$. *More precisely,* C *is a cycle of superattracting basins or the intersection* $S(f) \cap U_j$ *contains a point with infinite orbit.*

- *If* C *is a cycle of Siegel disks, then* $\partial U_j \subset P(f)$ *for all* $j \in \{0, \dots, p-1\}$.

If U *is a wandering domain of* f*, then all limit functions of* $\{f^n|_U\}$ *are constant and contained in the union* $(P(f))' \cup \{\infty\}$ *of the derived set* $(P(f))'$ *of* $P(f)$ *and* $\{\infty\}$.

For details, see [Ber93, Theorem 7], [BKL91, Lemma 2.1] and [BHK+93, Theorem].

Remark. We would like to comment on our treatment of the point at ∞. A perhaps more elegant way of incorporating polynomials and transcendental entire maps into the same setup, would have been the consideration of holomorphic maps on Riemann surfaces. But this would require introduction to a topic which, apart from this, is not tackled in this thesis. One could argue that if f is a polynomial of degree ≥ 2, then ∞ should be considered as a critical value and hence included into the set $C(f)$. But the fact is that even then, the consideration of the orbit of ∞ does not contribute to the understanding of the dynamics, since $f^{-1}(\infty) = \{\infty\}$. As Theorem 2.11 shows, the "finite" singular values determine the dynamics.

2.5 External addresses and dynamic rays

Throughout this paragraph, we will assume that f is transcendental entire.

Definition 2.12 (Tract, fundamental domain, static partition). Let $f \in \mathscr{B}$, let $D \supset S(f)$ be a Jordan domain and define $\overline{D}^c := \mathbb{C}\backslash\overline{D}$. A component of $G := f^{-1}(\overline{D}^c)$ is called a *tract* of f (with respect to D).
Suppose that there is a curve $\alpha \subset \overline{D}^c$ connecting ∂D to ∞ such that $\alpha \cap G = \emptyset$. A component of $f^{-1}(\overline{D}^c \setminus \alpha)$ is called a *fundamental domain* of f (with respect to D and α).
The set of all fundamental domains constructed with regard to such a Jordan domain D and a curve α will be called a *static partition* of f, and will usually be denoted by $\mathscr{S}(f, D, \alpha)$ or $\mathscr{S}$ (if f, D and α are implicitly known).

Remark. Our definition of a *tract* corresponds to what is usually called a *tract over* ∞ in the classification of singularities of the inverse function [Nev53, §XI.1].

Let D be a Jordan domain with $S(f) \subset D$. Then the restriction $f : T \to \overline{D}^c$ is a covering map for any tract T (with respect to D). The possible coverings of a punctured disk are well-known: T is either a punctured disk or a topological disk. Since f is not a polynomial and since it has no poles, every tract T of f is a simply-connected domain whose boundary ∂T is a Jordan arc tending to ∞ at both ends. (For details see [Eps93, p. 84].) For any such tract T, the map $f|_{\partial T}$ is a covering of ∂D. This implies that if B is any bounded domain then only finitely many tracts can intersect B: otherwise, there would exist a sequence of points z_i each of which belongs to the boundary of a tract T_i that intersects B, such that $f(\lim z_i) =: w \in \partial D$ is not evenly covered by f. In particular, the tracts have pairwise disjoint closures (in $\mathbb{C}$).
Now let us observe that for every $f \in \mathscr{B}$ there indeed exists a static partition: If D is a Jordan domain which contains $S(f)$ then $D' := f^{-1}(D) \setminus \overline{D}$ is open and each of its components has locally connected boundary in $\widehat{\mathbb{C}}$, hence any two boundary points of D' can be connected by a curve in D'. In particular,

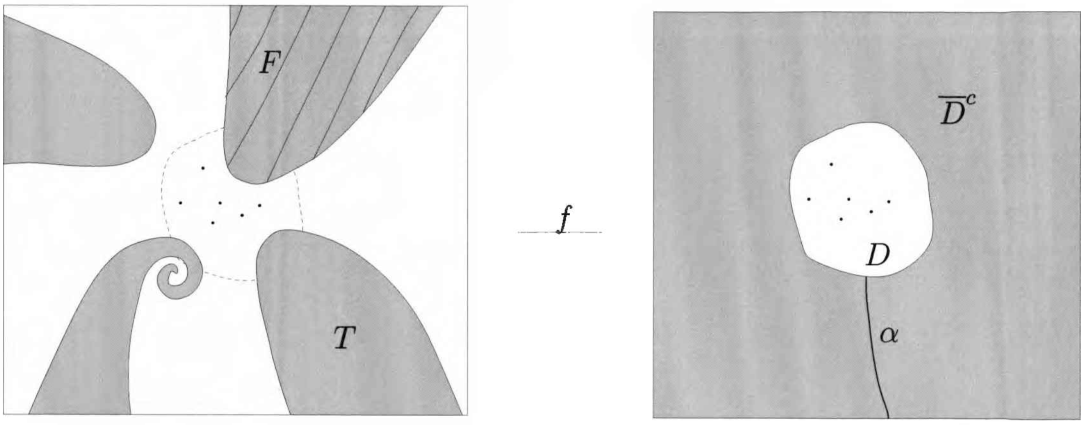

Figure 2.1: Tracts and fundamental domains of a function $f \in \mathscr{B}$.

since the tracts have pairwise disjoint closures, it follows that $\partial D \cap \partial D' \neq \emptyset$, so there exists a curve $\alpha \subset D'$ that connects ∂D and ∞ and hence induces a static partition of f.
Let $\mathscr{S}(f, D, \alpha)$ be a static partition. Since α is an unbounded curve in $\overline{D}^c$, its preimages are pairwise disjoint curves to ∞ that split the tracts into fundamental domains which are necessarily simply-connected. Furthermore, every fundamental domain is mapped by f conformally onto $\overline{D}^c \backslash \alpha$.

Remark. When dealing with a map f for which $P(f)$ is bounded, it sometimes makes sense to consider tracts with respect to a domain $D \supset P(f)$, since the fact that $S(f^n) \subset \bigcup_{i=0}^{n-1} f^i(S(f)) \subset P(f)$ implies that all inverse branches of any iterate of f can be defined at any point in $\overline{D}^c$.

Definition 2.13 (External address). Let $\mathscr{S} = \mathscr{S}(f, D, \alpha)$ be a static partition of f. If $z \in \mathbb{C}$ with $f^n(z) \in \overline{D}^c$ for all $n \geq 0$, then the *external address* $\operatorname{addr}(z) = \operatorname{addr}_{\mathscr{S}}(z)$ of z is the sequence $F_0 F_1 F_2 \ldots$ of fundamental domains in $\mathscr{S}$ defined by $f^n(z) \in F_n$.

Let $\mathscr{S} = \mathscr{S}(f, D, \alpha)$. Note that the fact that $\alpha \cap T = \emptyset$ for any tract T of f (w.r.t. D) guarantees that a point $z \in \overline{D}^c$ does indeed belong to a unique fundamental domain in $\mathscr{S}$. If $z \in I(f)$ then there exists an integer $n_0 \geq 0$ such that $|f^n(z)| \in \overline{D}^c$ for all $n \geq n_0$.
We will use $\underline{s}$ or $\underline{t}$ as standard notations for external addresses. It is not hard to see that if $\mathscr{S}$ and $\widetilde{\mathscr{S}}$ are two static partitions of f then the corresponding address spaces $\mathscr{S}^{\mathbb{N}}$ and $\widetilde{\mathscr{S}}^{\mathbb{N}}$ can be identified in a natural way. On every such

space $\mathscr{S}^{\mathbb{N}}$, we define the *one-sided shift map* σ in the usual way, i.e.,

$$\sigma : \mathscr{S}^{\mathbb{N}} \to \mathscr{S}^{\mathbb{N}}, \quad \underline{s} = F_0F_1F_2\ldots \mapsto \sigma(\underline{s}) := F_1F_2F_3\ldots\ .$$

Consequently, it makes sense to speak of external addresses that are periodic, preperiodic etc.

Definition 2.14 (Dynamic rays and ray tails). Let f be transcendental entire. A *ray tail* of f is an injective curve

$$g : [t_0, \infty) \to I(f)$$

(where $t_0 > 0$) such that for each $n \in \mathbb{N}$, $f^n|_g$ is injective, $\lim_{t\to\infty} f^n(g(t)) = \infty$ and such that, as $n \to \infty$, $f^n(g(t)) \to \infty$ uniformly in t.
A *dynamic ray* of f is then a maximal injective curve $g : (0, \infty) \to I(f)$ such that $g|_{[t_0,\infty)}$ is a ray tail for every $t_0 > 0$.

Example 2.15. Let $f(z) = \exp(z)$. Then every point in the set $\mathbb{R}_{\leq 0} := \{z : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$ converges uniformly to ∞ under iteration of f. But no iterated forward image of $\mathbb{R}_{\leq 0}$ is a curve to ∞, hence neither $\mathbb{R}_{\leq 0}$ nor any subset of it is a ray tail of f.
On the other hand, every interval $[a, \infty)$ with $a \in \mathbb{R}$ is a ray tail, hence none of the curves (a, ∞) is maximal. Thus the unique dynamic ray which contains all these tails is $(-\infty, \infty) = \mathbb{R}$.

If $\mathscr{S}$ is a static partition of the map f and g is a dynamic ray, then there exists $t_0 > 0$ such that for each $n \geq 0$, the curve $f^n(g([t_0, \infty)))$ is contained in a fundamental domain $F_n \in \mathscr{S}$. The sequence $\operatorname{addr}(g) = \operatorname{addr}_{\mathscr{S}}(g) = F_0F_1F_2\ldots$ is called the *external address of g (with respect to $\mathscr{S}$)*.

Definition 2.16 (Periodic rays). Let f be an entire function. A *periodic ray* of f is a maximal injective curve

$$g : (t_0, \infty) \to \mathbb{C}$$

such that $\lim_{t\to\infty} g(t) = \infty$ and such that there is an $n \geq 1$ with $g(2t) = f^n(g(t))$ for all $t > t_0 > 0$. The minimal such n is called the *period* of g.
As usual, we say that g *lands* at z_0 if $\lim_{t\to t_0} g(t) = z_0$.

Note that the dynamic ray from the previous example has a fixed external address. In fact, it is a fixed ray.

Proposition 2.17. *Let f be a transcendental entire map with bounded postsingular set. Then g is a periodic ray of f if and only if g is a dynamic ray of f with a periodic external address. Furthermore, no two distinct dynamic rays of f intersect.*

The equivalence between periodic rays and dynamic rays with periodic addresses is understood up to reparametrization. Note that the latter claim does not mean that two distinct dynamic rays must have distinct landing points.

Proof. It follows from the definitions that, after suitable reparametrization, any periodic ray is indeed a dynamic ray with a periodic external address.
Conversely, any dynamic ray g with $f^n(g) \subset g$ can be parametrized as a periodic ray in the sense of Definition 2.16. Hence it remains to show that a dynamic ray with a periodic external address is mapped into itself by an iterate of f. Let us assume that this is false, i.e., there is a static partition $\mathscr{S}$ of f and a dynamic ray $\tilde{g} \neq g$ with periodic external address $\operatorname{addr}_{\mathscr{S}}(g) = \operatorname{addr}_{\mathscr{S}}(\tilde{g})$. It follows from the proof of [Rem07a, Corollary 3.4] that $g \subset \tilde{g}$ or the other way around. (The given reference states only that g and $\tilde{g}$ intersect but the proof shows what we stated.) It then follows from [Rem07a, Lemma 3.3] that g equals $\tilde{g}$, contradicting our assumption. Hence we obtain the first claim of the proposition.
Now, let f be a transcendental entire map with a bounded postsingular set. Assume that the second claim is wrong, i.e., there exist two distinct dynamic rays g_1 and g_2 of f such that $g_1 \cap g_2 \neq \emptyset$. Let w denote an intersection point of these two rays and let $h_1 \subset g_1$ and $h_2 \subset g_2$ be any two ray tails containing w. Let $\mathscr{S}$ be a static partition. By definition, the iterates of f are converging uniformly to ∞ on h_1 and h_2, hence there is $n_0 \geq 0$ such that for every $n \geq n_0$, there exists a fundamental domain $F_n \in \mathscr{S}$ that entirely contains $f^n(h_1)$ and $f^n(h_2)$. As before, it follows from (the proof of) [Rem07a, Corollary 3.4] that $f^N(h_1) \subset f^N(h_2)$ or vice versa. By assumption, $P(f)$ does not intersect the orbit of h_1 and h_2. Hence, by pulling back, we obtain $f^k(h_1) \subset f^k(h_2)$ (or the other way around) for all $k \geq 0$. Since dynamic rays are maximal (in the sense that any tail is a ray tail), it follows that $g_1 = g_2$, contradicting the initial assumption. □

Next, we want to cite the result [Rem08, Theorem B.1] on landing behaviour of fixed rays.

Theorem 2.18 (Landing of fixed rays). *Let $f \in \mathrm{Hol}^*(\mathbb{C})$ and let $U \subset \mathbb{C}$ a hyperbolic domain such that $U \subset f(U)$ and the restriction $f : U \to f(U)$ is a covering map. Furthermore, suppose that $f|_U$ is not an irrational rotation of a disk, punctured disk or annulus.*
Let $\gamma : (-\infty, 1] \to U$ be a curve with $f(\gamma(t)) = \gamma(t+1)$ for all $t \leq 0$. Then every accumulation point of $\gamma(t)$ in $\overline{U}$ is a fixed point of f.

This has the following consequence for entire maps with bounded postsingular set ([Rem08, Corollary B.4]).

Corollary 2.19. *Let f be a transcendental entire map for which $P(f)$ is bounded. Then every periodic ray of f lands at a repelling or parabolic periodic point of f.*

Clearly, the period of the landing point divides the period of the ray.

Proof. Let g be a periodic ray of f. Hence g is a fixed ray of some iterate f^n. Denote by V the unique unbounded component of $\mathbb{C} \setminus P(f)$ and let U be the unique component of $(f^n)^{-1}(V)$ that contains g. The restriction $f^n : U \to V$ is a covering map, since $S(f^n) \subset P(f) \subset \mathbb{C} \setminus V$. By Lemma 2.4, V is hyperbolic and so is U. Furthermore, $f^n(P(f)) \subset P(f)$ implies that $U \subset V$. We can now reparametrize g such that it has the form of the curve γ in Theorem 2.18. The claim now follows from Theorem 2.18 and the fact that g does not land at ∞ [Rem08, Theorem B.2]. □

Finally, let us state an important result on the existence of dynamic rays that we will use frequently. For details and proofs see [RRRS, Theorems 1.1, 4.2] and [Rem08, Theorem 2.4].

Theorem 2.20. *Let $f \in R^3S$. Then every escaping point of f is either on a dynamic ray or the landing point of a dynamic ray. Furthermore, for every periodic external address $\underline{s}$ there exists a (periodic) dynamic ray with address $\underline{s}$.*

3 Functions with bounded singular sets

In this section, we present the classes of maps that are of main interest to us. They will be characterized by certain relations between (post)singular values and the Fatou and Julia set. The following notations will be used frequently.

$$S_{\mathcal{F}} := S(f) \cap \mathcal{F}(f) \quad \text{and} \quad S_{\mathcal{J}} := S(f) \cap \mathcal{J}(f),$$
$$P_{\mathcal{F}} := P(f) \cap \mathcal{F}(f) \quad \text{and} \quad P_{\mathcal{J}} := P(f) \cap \mathcal{J}(f).$$

It will turn out that the Fatou sets of the functions which we will study consist of attracting or parabolic basins (or are empty). In many situations, we will need to remove compact subsets from such components such that the established domains are topologically simple and have nice mapping properties; this is the context of Section 3.1.

If not stated differently, we will assume throughout Section 3 that the considered maps belong to $\mathrm{Hol}^*(\mathbb{C})$; the maps defined in Sections 3.2, 3.3 —geometrically finite and subhyperbolic maps —are additionally assumed to be transcendental entire. The reason is that both terms already exist for rational maps but the definitions in both cases are not equivalent.

3.1 Compact subsets of attracting and parabolic basins

Recall that $\mathcal{A}(f)$ and $\mathcal{P}(f)$ denote the sets of all points that converge to an attracting or parabolic cycle of f, respectively (where the parabolic cycles themselves do not belong to $\mathcal{P}(f)$). It follows from Theorem 2.10 that $\mathcal{A}(f), \mathcal{P}(f) \subset \mathcal{F}(f)$. Also, if $C \subset \mathcal{A}(f)$ is a compact set, then $\overline{O^+(C)} := \overline{\bigcup_{n\geq 0} f^n(C)}$ is again a compact set which belongs to $\mathcal{A}(f)$. Likewise, if $C \subset \mathcal{P}(f)$ is compact then $\overline{O^+(C)}$ is a compact subset of $\mathcal{P}(f) \cup \mathrm{Par}(f)$.

Proposition 3.1. *Let $f \in \mathrm{Hol}^*(\mathbb{C})$ and let $C \subset \mathcal{A}(f)$ be a compact set. Then there exist Jordan domains $D_1, \dots, D_n$ such that if $D := \bigcup_{i=1}^n D_i$ then $f(D) \Subset D \Subset \mathcal{A}(f)$ and $U_\varepsilon(\overline{O^+(C)}) \subset D$ for some $\varepsilon > 0$.*

Proof. The components of $\mathcal{A}(f)$ form an open cover of $\widetilde{C} := \overline{O^+(C)}$. Since $\widetilde{C}$ is compact, there are finitely many components $A_1, \dots, A_n$ of $\mathcal{A}(f)$, uniquely determined such that their union covers $\widetilde{C}$ and $\widetilde{C}$ has nonempty intersection with each A_i. For every i, we define $\widetilde{C}_i := \widetilde{C} \cap A_i$. Since $f(\widetilde{C}) \subset \widetilde{C}$, at least one of the domains A_i must be periodic, which in this case means that it is one of the immediate attracting basins of some attracting cycle Z. Furthermore, $A^*(Z) \subset \bigcup_{i=1}^n A_i$. We can assume w.l.o.g. that $Z := \{z_1, \dots, z_m\}$ is the only attracting cycle that intersects $\widetilde{C}$; otherwise, we can repeat the procedure independently for any other attracting cycle. Observe that by assumption, every z_i belongs to $\widetilde{C}$. Let $A_1, \dots, A_m$ be the immediate attracting basins of the points $z_1, \dots, z_m$, respectively.

Let us first look at the domains $A_{m+1}, \dots, A_n$ that are not periodic. We can assume for simplicity that $A_n \xrightarrow{f} A_{n-1} \xrightarrow{f} \dots \xrightarrow{f} A_{m+1} \xrightarrow{f} A_m$, since if this is not the only non-periodic chain, we can repeat the same procedure independently for any such chain of attracting basins. The set $\widetilde{C}_n$ has positive distance to ∂A_n, hence there is $\varepsilon_n > 0$ such that $J_n := U_{\varepsilon_n}(\widetilde{C}_n) \Subset A_n$. For $n-1$, we choose $\varepsilon_{n-1} > 0$ such that $J_{n-1} := U_{\varepsilon_{n-1}}(\widetilde{C}_{n-1} \cup \overline{f(J_n)}) \Subset A_{n-1}$. We proceed successively until we have constructed J_{m+1}.

Let us now consider an immediate attracting basin A_i, i.e., $1 \le i \le m$. It is mapped by f^m into itself since z_i is a fixed point of f^m. $\widetilde{C}_i$ is a compact subset of A_i, hence there is a neighbourhood J_i of $\widetilde{C}_i$ which is compactly contained in A_i. If $i = m$ we additionally assume that J_m contains a neighbourhood of $\overline{f(J_{m+1})}$. Let U_i be a linearizing neighbourhood of z_i. Now, since $\overline{J_i}$ is a compact set and since all points in A_i converge to z_i under iteration of f^m, there is an integer $N \ge 0$ such that $\overline{f^{mN}(J_i)} \subset U_i$. Hence

$$\bigcup_{j=0}^{N-1} \overline{f^{mj}(J_i)} \cup \overline{U_i} =: \widetilde{K}_i$$

is a compact connected set that contains a neighbourhood of $\widetilde{C}_i$ in its interior. ($\widetilde{K}_i$ is connected since J_i contains z_i and so do all its images.) By construction, $\widetilde{K}_i$ is mapped into its interior by f^m. So in particular, $\overline{O^+(\widetilde{K}_i)} = O^+(\widetilde{K}_i) = \bigcup_{i=0}^{m-1} f^i(\widetilde{K}_i)$.

For every i we define a new set

$$\widetilde{\widetilde{K}}_i := \bigcup_{j=1}^{m} O^+(\widetilde{K}_j) \cap A_i.$$

By construction, for all $i \in \{1, \dots, m\}$ and $k \in \{m+1, \dots, n\}$ the sets $\widetilde{\widetilde{K}}_i$ and

$\overline{J_k}$ are compact and connected. Furthermore, their union $\tilde{\tilde{K}} := (\bigcup_{i=1}^m \tilde{\tilde{K}}_i) \cup (\bigcup_{k=m+1}^n \overline{J_k})$ is mapped compactly into its interior by f.
Note that the components of $\tilde{\tilde{K}}$ are not necessarily simply-connected. Let B be a bounded component of its complement. Since B and $f(B)$ are bounded, it follows from the Open Mapping Theorem that $\partial f(B) \subset f(\partial B) \subset \tilde{\tilde{K}}$. Define K to be the fill-in of $\tilde{\tilde{K}}$ (recall that this is the union of $\tilde{\tilde{K}}$ and the bounded components of its complement). Then K is a full set whose interior contains $f(K)$ and a neighbourhood of $\tilde{C}$. For every component K_i of K, we choose D_i to be a Jordan domain in K_i that contains $\tilde{C}_i$, and the claim follows. □

For compact subsets of parabolic basins we obtain an analogous statement.

Proposition 3.2. *Let $f \in \mathrm{Hol}^*(\mathbb{C})$ and let $C \subset \mathcal{P}(f)$ be a compact set. Then there exist simply-connected domains $D_1, \dots, D_n$ such that if $D := \bigcup_{i=1}^n D_i$ then $f(D) \subsetneq D$, $\overline{O^+(C)} \subsetneq D \cup \mathrm{Par}(f)$ and $\overline{D} \setminus \mathrm{Par}(f) \subset \mathcal{P}(f)$.*

Proof. The components of $\mathcal{P}(f)$ form an open cover of the compact set C, hence there is a finite subcover. Any of the components of this finite subcover is preperiodic, hence there are finitely many components $P_1, \dots, P_n$ of $\mathcal{P}(f)$ unique with the property that their union covers $\tilde{C} := O^+(C)$ and $\tilde{C}$ has nonempty intersection with every P_i. For every i, define $\tilde{C}_i := \tilde{C} \cap P_i$. Observe that $\tilde{C}_i$ is compact if P_i is not a periodic component. If P_i is periodic then the union of $\tilde{C}_i$ and the unique parabolic periodic point in ∂P_i is a compact set.
As in the previous proof, we can assume w.l.o.g. that only one parabolic periodic cycle intersects $\tilde{C}$. Assume that this cycle is a fixed point, say q, of multiplicity $m+1$, where $m > 0$; the periodic case is analogous. By the Parabolic Flower Theorem [Mil06, Theorem 10.7] there are m attracting petals of arbitrarily small diameter attached to q. Let $A_1, \dots, A_m$ be such a collection of petals at q. We order the domains P_i such that for every $1 \le i \le m$, the petal A_i is a subset of P_i. Observe that $\tilde{C}$ intersects every petal A_i. There is a positive number δ such that if $|f^n(z) - q| \le \delta$ holds for all n, then z is contained in some A_i [EL90, Chapter 1, §3.3].
As before, it is no loss of generality to assume that the domains $P_{m+1}, \dots, P_n$ satisfy $P_n \xrightarrow{f} P_{n-1} \xrightarrow{f} \dots \xrightarrow{f} P_{m+1} \xrightarrow{f} P_m$. Since the intersection of $\tilde{C}$ with those components is indeed compact, we can find for every $i \in \{m+1, \dots, n\}$ a neighbourhood $J_i \Subset P_i$ of $\tilde{C}_i$ such that $f(J_n) \Subset J_{n-1}, \dots, f(J_{m+2}) \Subset J_{m+1}$. We omit the details since there is no difference to the previous proof.

Now let $i \in \{1, \dots, m\}$. Then $f^m(P_i) \subset P_i$. The set $\widetilde{C}_i \setminus A_i$ is a compact subset of P_i, hence it has a neighbourhood $J_i \Subset P_i$. Furthermore, we can assume that J_i contains for at least one point $z \in J_i$ also its iterated image $f^m(z)$. If $i = m$ we assume additionally that J_m contains $\overline{f(J_{m+1})}$.
There exists an N which is minimal with the property that $\overline{f^n(J_i)}$ is contained in the δ-neighbourhood of q for all $n > N$. We now define the set $\widetilde{K}_i$ to be $\bigcup_{j=0}^{N} \overline{f^{mj}(J_i)} \cup \overline{A_i}$. Clearly, every $\widetilde{K}_i$ is compact and connected. Also note that $\widetilde{K}_i \cap \mathcal{J}(f) = \{q\}$. From here, the steps are the same as in the proof of the previous proposition. □

3.2 Geometrically finite maps

Let us start with the first class of functions that are of interest to us.

Definition 3.3. A transcendental entire map f is called *geometrically finite* if $P_{\mathcal{J}}$ is finite and $S_{\mathcal{F}}$ is compact.

Note that every such map belongs to the Eremenko-Lyubich class $\mathscr{B}$. Furthermore, since $S(f^n) \subset \bigcup_{i=0}^{n-1} f^i(S(f))$ and $\mathcal{F}(f^n) = \mathcal{F}(f)$, it follows that every iterate of a geometrically finite map is again geometrically finite.
The notion of geometrically finite *rational* maps already exists. Following McMullen [McM00], a rational function R is called geometrically finite if $P(R) \cap \mathcal{J}(R)$ is finite. Using classical results on dynamics of rational maps, one can easily deduce that the Fatou set of such a map is the union of finitely many attracting and parabolic basins [McM00, Chapter 6]. The following statement shows that the same holds for a geometrically finite transcendental entire map.

Proposition 3.4. *Let f be a geometrically finite (transcendental entire) map. Then the Fatou set of f is either empty or consists of finitely many attracting or parabolic basins. Furthermore, every periodic cycle in $\mathcal{J}(f)$ is repelling or parabolic.*

Proof. First note that f cannot have wandering domains. Indeed, if W was a wandering domain, then all limits of orbits of points in W would belong to $P_{\mathcal{J}}$ [Theorem 2.11], and hence the iterates in W would converge locally uniformly to a single periodic orbit in $P_{\mathcal{J}}$. This orbit clearly cannot be repelling or parabolic. By a result of Perez-Marco [PM97], this orbit also cannot be irrationally indifferent. Hence f has no wandering domains. Additionally, if

f had a Siegel disk, then its boundary would be contained in $P(f)$ [Theorem 2.11]. This is again impossible because $P_{\mathcal{J}}$ is finite, so f has no Siegel disks. Thus the Fatou set is the union of attracting and parabolic basins.
The set of attracting and parabolic basins forms an open cover of the compact set $S_{\mathcal{F}}$. Hence there exist finitely many attracting and parabolic basins that cover this set. On the other hand, every attracting or parabolic basin must contain at least one point of $S(f)$ [Theorem 2.11]. This proves the first claim.
Furthermore, if z_0 was an irrationally indifferent periodic point in $\mathcal{J}(f)$, then there would be a sequence w_k of points in $P(f)$ converging nontrivially to z_0 [Mil06, Corollary 14.4]. Since $P_{\mathcal{J}}$ is finite and $P_{\mathcal{F}}$ is contained in the union of finitely many attracting and parabolic basins, this is impossible. □

Remark. Let f be geometrically finite. Then $P(f)$ is bounded. Furthermore, every critical point in $\mathcal{J}(f)$ is strictly preperiodic and every singular value of f in $\mathcal{J}(f)$ is eventually mapped to a repelling or parabolic periodic cycle.

3.3 Subhyperbolic maps

In Chapter 6 we will consider transcendental entire maps that are *strongly subhyperbolic* and show that the Julia set of every such map f can be described as a quotient of the Julia set of a map g that is dynamically much simpler than f; the map g will be a *nonescaping-hyperbolic map* with connected Fatou set.
Let us start with the definition of a subhyperbolic map.

Definition 3.5. A transcendental entire function f is called *subhyperbolic* if $P_{\mathcal{J}}$ is finite and $P_{\mathcal{F}}$ is compact.

Hence subhyperbolic maps are exactly those geometrically finite maps that have no parabolic cycles.
Note that every *postsingularly finite* map, i.e., a map with finite postsingular set, is subhyperbolic. The Fatou set of such a map is either empty or the union of finitely many superattracting basins.
Again, the notion of subhyperbolic *rational* maps already exists: A rational map is said to be subhyperbolic if it is *expanding* with respect to a Riemannian metric which has a discrete set of "mild" singularities — a so-called *orbifold metric.* (The precise definition of orbifolds and orbifold metrics will be given in 6.1.) By [Mil06, Theorem 19.6], this is equivalent to the requirement that every

singular value has finite orbit or converges to an attracting cycle on the sphere! As explained in Section 2.4, a polynomial can be considered as a holomorphic map on the sphere with ∞ as a superattracting fixed point, hence there are polynomials that are subhyperbolic in the rational sense but not in the sense of Definition 3.5. An example is given by $p : z \mapsto z^2 + 1$ since the unique singular value 1 belongs to $I(p) \subset \mathcal{F}(p)$.
However, for an *arbitrary* subhyperbolic transcendental map it is not possible to define an orbifold metric on a neighbourhood of the Julia set that is expanded by the map. Here we present those subhyperbolic transcendental maps for which, as we will prove later, this will be possible. The mentioned expansion property turns out to be crucial for the proof of the main result in Section 6.

Definition 3.6. A subhyperbolic (transcendental entire) map f is called *strongly subhyperbolic* if $\mathcal{J}(f) \cap A(f) = \emptyset$ and there is a constant $R < \infty$ such that $\deg(f, z) < R$ holds for all $z \in \mathcal{J}(f)$.

3.4 Hyperbolic maps

As already mentioned, *hyperbolic* functions will play a great role in the following chapters. In fact, all major results in this thesis will apply to those hyperbolic maps that are transcendental entire.

Definition 3.7. A map f is called *hyperbolic* if $P_{\mathcal{J}} = \emptyset$ and $S_{\mathcal{F}}$ is compact.

Note that, unlike in the case of geometrically finite and subhyperbolic maps, the definition of hyperbolicity addresses not only transcendental entire maps but also polynomials. Also note that every map $f \in \mathrm{Hol}^*(\mathbb{C})$ that is hyperbolic, automatically belongs to $\mathrm{Hol}^*_b(\mathbb{C})$.

Definition 3.8. An entire function is called *nonescaping-hyperbolic* if $P_{\mathcal{J}} = \emptyset$ and $P_{\mathcal{F}}$ is compact.

Let us sum up some statements on hyperbolic maps. We will omit a proof since the relations follow immediately from Proposition 3.4 and the Theorems 2.10, 2.11.

Proposition 3.9. *Let $f \in \mathrm{Hol}^*(\mathbb{C})$. If f is nonescaping-hyperbolic, then $\mathcal{F}(f) = \mathcal{A}(f) \neq \emptyset$ and* $\mathrm{Attr}(f)$ *is finite. If f is transcendental entire, then the following statements are equivalent.*

(*i*) *f is hyperbolic.*

(*ii*) *f is nonescaping-hyperbolic.*

(*iii*) *f is geometrically finite and $P_{\mathcal{J}} = \emptyset$.*

(*iv*) *$S(f)$ is bounded and every point in $S(f)$ belongs to an attracting basin.*

If f is a polynomial then (ii) $\Leftrightarrow$ (iv). If f is a polynomial that satisfies (i), then (ii) does not hold if and only if $C(f) \cap I(f) \neq \emptyset$. In this case, $\mathcal{F}(f) = \mathcal{A}(f) \cup I(f)$.

Observe that the polynomial $p : z \mapsto z^2 + 1$ (see previous paragraph) is hyperbolic but $P_{\mathcal{F}}$ is not bounded.
Formulated for rational maps, Definition 3.7 is equivalent to the classical definition by which a rational map is hyperbolic if it expands a *conformal* Riemannian metric on a neighbourhood of its Julia set [Mil06, Theorem 19.1]. For a hyperbolic transcendental entire map f, one can give a similar description: By Proposition 3.1, there exists a bounded open set D such that $(f(D) \cup P(f)) \Subset D \Subset \mathcal{F}(f)$. Hence $\mathbb{C} \setminus \overline{D} =: U$ is a neighbourhood of $\mathcal{J}(f)$, and the map $f : f^{-1}(U) \to U$ is then a covering map which *uniformly expands* the hyperbolic metric on U [Rem, Lemma 5.1].
We would like to remark that the requirement in Proposition 3.9 (*iv*) that $S(f)$ is bounded is crucial in order to achieve that $\mathcal{F}(f) = \mathcal{A}(f)$ holds: As Example D in [KS08] shows, there is a transcendental entire map f with wandering domains for which all singular values are mapped by f to an attracting fixed point.
We will consider nonescaping-hyperbolic maps in Chapter 7. Note that we can apply the tools from Paragraph 3.1 to the (post)singular set of every such map.

3.5 Functions of disjoint type

Definition 3.10. A hyperbolic transcendental entire map f is said to be of *disjoint type* if $\mathcal{F}(f)$ is connected.

Proposition 3.11. *The following statements are equivalent for a transcendental entire map f:*

1. f is of disjoint type.

2. *f has a unique attracting fixed point and $P(f)$ is a compact subset of its immediate basin of attraction.*

3. *There exists a (bounded) Jordan domain $D \supset S(f)$ such that $\overline{f(D)} \subset D$.*

Proof. Let f be a transcendental entire map of disjoint type. In particular, f is nonescaping-hyperbolic. By Definition 3.8, $P(f)$ is a compact subset of $\mathcal{F}(f)$. Since $\mathcal{F}(f)$ is connected, $P(f)$ must be a compact subset of a completely invariant component of $\mathcal{F}(f)$, which can only be the immediate attracting basin of an attracting fixed point of f, showing that (1) implies (2).
It follows immediately from Proposition 3.1 that (2) implies (3).
To see that (3) implies (1), let us choose a domain $D \supset S(f)$ such that $\overline{f(D)} \subset D$. By Montel's Theorem, D is contained in a component of $\mathcal{F}(f)$, so, in particular, f is hyperbolic. By Theorem 2.11 every immediate attracting basin contains at least one singular value, hence f has a unique attracting cycle, which is a fixed point contained in D. Hence every point $z \in \mathcal{F}(f)$ is eventually mapped into D, showing that $\mathcal{F}(f) = \bigcup_{n \geq 0} f^{-n}(D)$. On the other hand, for every $n \geq 1$, $f^{-n}(D)$ is connected and $f^{-(n+1)}(D) \supset f^{-n}(D)$, hence $\mathcal{F}(f)$ is connected. □

Disjoint type maps are quite well understood. Recall that by Theorem 2.10, the Fatou set of such a map is simply-connected. We present two "known" statements on topology of Julia sets of such functions.

Theorem 3.12. *Let g be of a disjoint type. Then $I(g)$ is disconnected.*

Proof. By Proposition 3.11 there is a static partition $\mathscr{S} = \mathscr{S}(g, D, \alpha)$ with $\overline{g(D)} \subset D$. Hence $I(g)$ is contained in the union of the fundamental domains in $\mathscr{S}$.
By Theorem 2.7, there exists a point $w \in I(g)$, so every fundamental domain must intersect $I(g)$ since it contains a preimage of w. Let $\tilde{F} \in \mathscr{S}$ and let U denote the union of the sets in $\mathscr{S} \setminus \{\tilde{F}\}$. Then U and $\tilde{F}$ are two disjoint nonempty open sets whose union covers $I(g)$. Thus $I(g)$ is disconnected. □

Theorem 3.13. *Let $g \in R^3S$ be of disjoint type. Then $\mathcal{J}(g)$ is a Cantor bouquet.*

A *Cantor bouquet* is a subset of $\mathbb{C}$ that is homeomorphic to a *straight brush* in the sense of Aarts and Oversteegen [AO93, Definition 1.2]. Roughly, we should

think of a Cantor bouquet as a union of uncountably many pairwise disjoint curves, each of which connects a distinguished point in the plane to ∞.
For disjoint type maps of finite order, Theorem 3.13 was first established by Barański [Bar07]. In [RRRS, Theorem 4.7], the authors show that the Julia set of a disjoint type map $g \in R^3S$ is an "absorbing brush". As it is already remarked in [RRRS], it is fairly obvious — e.g. using the topological characterization in [AO93, Theorem 3.11] — that an absorbing brush is actually homeomorphic to a straight brush. However, we will not state the details here since it would require a large amount of background material which is of no further use in this thesis.

3.6 Relations between the respective classes

When we restrict to transcendental entire maps, then the classes we introduced in Section 3 are related to each other as follows.

$$\{\text{disjoint type}\} \subsetneq \{\text{(nonescaping-)hyperbolic}\} \subsetneq \{\text{strongly subhyperbolic}\} \subsetneq \{\text{subhyperbolic}\} \subsetneq \{\text{geometrically finite}\} \subsetneq \{\text{bounded postsingular set}\} \subsetneq \mathscr{B}$$

The stated inclusions are indeed strict:
The map $f_1 : z \mapsto \frac{\pi}{2} \sin z$ is hyperbolic, since both critical values $\pm\pi/2$ of f_1 are superattracting fixed points. By Proposition 3.11, f_1 is not of disjoint type.
Let us consider $f_2 : z \mapsto \pi \sinh z$. This function has no asymptotic values and the only critical values are $\pm\pi i$. Both are mapped by f_2 to the repelling fixed point 0, hence f_2 is strongly subhyperbolic. But f_2 is not hyperbolic since the critical values are in the Julia set. This example will be investigated in Section 6.5.
The function $f_3 : z \mapsto 2\pi i\, e^z$ is subhyperbolic since the unique singular value 0 is prefixed. On the other hand, 0 is an asymptotic value and since it is strictly preperiodic, it belongs to $\mathcal{J}(f_3)$. Hence f_3 is not strongly subhyperbolic. We will consider a different type of examples of subhyperbolic but not strongly subhyperbolic maps in Section 6.6.
Let us now look at the map $f_4 : z \mapsto e^{z-1}$. We see that 1 is a parabolic fixed point of f_4 and the unique singular value 0 converges to 1. Hence $S(f_4) = \{0\}$

is a compact subset of $\mathcal{F}(f_4)$ and $P(f_4) \cap \mathcal{J}(f_4) = \{1\}$ is indeed finite, implying that f_4 is geometrically finite. But clearly, f_4 is not subhyperbolic, since it has a parabolic fixed point.

An example of a function that is not geometrically finite but has bounded postsingular set is the function

$$f_5(z) = \frac{12\pi^2}{5\pi^2 - 48}\left(\frac{(\pi^2-8)z + 2\pi^2}{z(4z-\pi^2)}\cos\sqrt{z} + \frac{2}{z}\right).$$

This map, which was introduced in [Ber02], has the property that the asymptotic value 0 is a parabolic fixed point and at the same time the accumulation point of critical values that lie in the basin of 0. Hence f_5 is not geometrically finite. But $S(f_5)$ is bounded and every singular value converges to 0, hence $P(f_5)$ is bounded. We will examine this function in Section 4.4; it is an example of a non-geometrically finite map for which, as we will prove, Theorem 1.1 is still true.

At last, it is obvious that there are maps in class $\mathscr{B}$ which do not have a bounded postsingular set: for instance, the exponential map $f_6 : z \mapsto e^z$ is in class $\mathscr{B}$ but $P(f_6)$ is unbounded since the singular value 0 escapes to ∞.

4 Periodic points are landing points

In this chapter, we will consider periodic points that lie in the Julia set and show that — if the given map is a geometrically finite function in the class R^3S — every such point is the landing point of a periodic ray. This will follow from the next result.

Theorem 4.1. *Let f be a transcendental entire function, z_0 a repelling or parabolic fixed point of f and assume that there is an admissible expansion domain U of f at z_0. If for any periodic external address $\underline{s}$ there exists a periodic ray of f with address $\underline{s}$, then there is a periodic ray of f landing at z_0.*

Admissible expansion domains will be defined in Section 4.1; roughly, these will be hyperbolic domains with simple topology, on which the map f is expanding. We will show that every (iterate of a) geometrically finite map has an admissible expansion domain at every fixed point in the Julia set.

In the case of geometrically finite exponential maps, our theorem is due to Schleicher and Zimmer [SZ03] (although in the case where the singular value is preperiodic some of the details are only sketched). This was extended to cosine maps with strictly preperiodic critical values in [Sch07a]. The general strategy of our proof, which will be described now, follows the same idea as these papers. By passing to a suitable iterate, we can assume that the considered repelling or parabolic periodic point z_0 is a fixed point. The idea is to start with *any* given curve connecting z_0 to infinity, and pull back this curve using iterates of the map f. Using hyperbolic contraction arguments, we prove that this procedure yields only finitely many different curves up to homotopy. This then allows us to associate a periodic external address to each of these curves. From Theorem 2.20 we then know that there exists a periodic ray corresponding to this address. It will then follow, again by hyperbolic contraction, that this ray lands at z_0.

Since the "ad-hoc" method that was used to obtain hyperbolic contraction estimates in [SZ03, Sch07a] appears to be difficult to adapt to our more general setting, we develop a rather natural construction using hyperbolic geometry.

We would like to emphasize that the case when $P(f)$ is finite requires less technical constructions than the general setting. This is why we consider this special case separately in the proofs of some results required for the proof of Theorem 4.1.

Structure of Chapter 4

Sections 4.1 - 4.3 address geometric constructions based on hyperbolic plane geometry; we will require most of the content from Section 2.2. In Section 4.4 we will show that our geometric setup is admissible for any iterate of a geometrically finite map. We will also give an example of a map which is not geometrically finite but to which our methods still apply. The last section focuses on the final steps for the proof of Theorem 4.1.

If not stated differently, we will assume throughout Chapter 4 that the considered maps are transcendental entire.

4.1 Admissible expansion domains

To prove Theorem 4.1 we will use a hyperbolic domain U such that our function f is expanding with respect to the hyperbolic metric of U, and such that U has simple topology. Our requirements are formalized in the following definition.

Definition 4.2 (Admissible expansion domain). Let f be a transcendental entire function and let z_0 be a fixed point of f which is either repelling or parabolic. A domain $U = U(f, z_0) \subset \mathbb{C}$ is called an *admissible expansion domain* of f at z_0, if the following properties hold:

(*a*) $U \subset \mathbb{C} \setminus (P(f) \cup \{z_0\})$ and ∞ is an isolated point of $\hat{\partial} U$.

(*b*) $f^{-1}(U) \subsetneq U$.

(*c*) U is finitely-connected. Furthermore, $U \neq \mathbb{C} \setminus P(f)$.

(*d*) The point $z_0 \in \partial U$ is accessible from U. If K_0 is the component of $\mathbb{C} \setminus U$ containing z_0, then $K_0 \setminus \{z_0\}$ has finitely many components.

Note that every map for which there is an admissible expansion domain at some repelling or parabolic fixed point must have a bounded postsingular set. Furthermore, it follows from Definition 4.2 and Theorem 2.11 that, if $z_0 \in \mathbb{C}$ is a puncture of U, then z_0 is a repelling fixed point of f. We will show later that, if $P(f)$ is finite, then we can choose $\mathbb{C} \setminus U$ to be finite as well.

Also note that if U is an admissible expansion domain of f at z_0, then U is also an admissible expansion domain of f^n at z_0.

Let U be an admissible expansion domain of f. Observe that by Lemma 2.4, U is hyperbolic. By definition, there exists at least one point $w \in \mathbb{C} \setminus (U \cup P(f))$. Recall that $w \notin S(f)$ implies that w is not an exceptional value, hence w has infinitely many preimages under f.

Standing assumption. Throughout Section 4.2 and 4.3 we will assume that f is a transcendental entire map and z_0 is a repelling or parabolic fixed point of f with an admissible expansion domain, denoted by U.

It will become clear that the existence of admissible expansion domains is what is essential for our idea to work.

4.2 Legs and the leg map $\mathscr{L}$

Definition 4.3. A *leg* is an injective curve $\gamma : [0, \infty] \to U \cup \{z_0, \infty\}$ such that

(i) $\gamma|_{(0,\infty)} \subset U$,

(ii) $\gamma(0) = z_0$ and $\gamma(\infty) = \infty$.

Two legs γ_1 and γ_2 are called *equivalent* ($\gamma_1 \sim \gamma_2$) if they are homotopic in U relative to the set of endpoints $\{z_0, \infty\}$. For a leg γ we will denote its equivalence class by $[\gamma]$.

By assumption, z_0 is not a critical point of f, so every leg ending at z_0 has a unique preimage curve ending at z_0 and this is again a leg. The map which assigns such a pullback to each leg γ will be called the *leg map* and denoted by $\mathscr{L}$. As usual, we will denote the n-th iterate of $\mathscr{L}$ by $\mathscr{L}^n$.

It follows from the Homotopy Lifting Property that if $\gamma_1 \sim \gamma_2$, then this also holds for their images, i.e., $\mathscr{L}(\gamma_1) \sim \mathscr{L}(\gamma_2)$. Hence, *the leg map $\mathscr{L}$ descends to a map on the set of equivalence classes of legs.*

We will often replace pieces of arbitrary legs by pieces of geodesics in their homotopy classes, which is possible by Proposition 2.2. We will call a leg that is a geodesic with respect to the hyperbolic metric on U a *geodesic leg.*

We are now able to formulate the main result of Section 4.1.

Theorem 4.4. *Let γ be a leg. Then there exist natural numbers $m < n$ such that $\mathscr{L}^m(\gamma) \sim \mathscr{L}^n(\gamma)$.*

The proof of this theorem will be given at the end of Section 4.3.

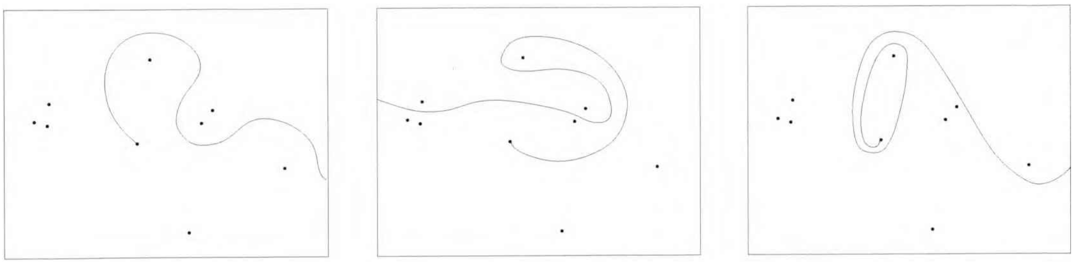

Figure 4.1: Legs belonging to three different equivalence classes.

4.3 Iteration of $\mathscr{L}$ and a finiteness statement

From now on, let us denote the density of the hyperbolic metric on the admissible expansion domain U of f at z_0 by $\rho_U(z)$. We define

$$V := f^{-1}(U).$$

Note that V does not have to be connected. Every component V_i of V is again a hyperbolic domain with corresponding density map ρ_{V_i}. If $z \in V$, then z lies in a unique component V_i and for simplicity, we will denote the density of the hyperbolic metric at z by $\rho_V(z)$.

Proposition 4.5. *Let $Y \subset U$ be a compact connected set. Then there exists a constant $\eta < 1$ such that*

$$\rho_U(z) < \eta \cdot \rho_V(z) \quad \textit{holds for all } z \in f^{-1}(Y).$$

Proof. Recall that $f(\partial V) \subset \partial U$ and $V \subsetneq U$. So for any $R > 0$, the set $K_R := f^{-1}(Y) \cap \{z : |z| < R\}$ is a compact subset of V. By Pick's theorem, for any $R > 0$ there exists a constant $\eta_R < 1$ such that $\rho_U(z) < \eta_R \cdot \rho_V(z)$ for all $z \in K_R$. Hence we have to consider only sufficiently large points $\tilde{z} \in f^{-1}(Y)$. Let $w \in \mathbb{C} \setminus (U \cup P(f))$ be a non-exceptional value of f. Then w has infinitely many preimages under f and all but finitely many of them are contained in $U \backslash V$.

Claim. There exists a sequence $w_j \in U \backslash V$ and a constant $K > 1$ such that $|w_{j+1}| \leq K|w_j|$ and $f(w_j) = w$ holds for all $j \in \mathbb{N}$.

Proof of claim. A sketch can be found in the proof of [Rem, Lemma 5.1]. For completeness we will elaborate the arguments given in [Rem].

Let $\gamma \subset \mathbb{C}$ be a Jordan curve, such that the bounded component of $\mathbb{C} \backslash \gamma$ contains $S(f)$ but not w, and let U_∞ denote the unbounded component of $\mathbb{C}\backslash\gamma$. Then $f^{-1}(U_\infty)$ is a countable union of tracts T_i and $f|_{T_i} : T_i \to U_\infty$ is a universal covering for every i. Let us pick a tract T_0. Since $w \in U_\infty$, there is an infinite sequence w_i of preimages of w in T_0, such that the distance $d_{T_0}(w_i, w_{i+1})$ measured in the hyperbolic metric of T_0 is constant.

We can assume w.l.o.g. that $0 \notin T_0$, and so by equation (2.1) we obtain

$$\rho_{T_0}(z) \geq \frac{1}{2|z|}.$$

Let $A := d_{T_0}(w_i, w_{i+1})$. It follows that

$$A = \inf_\gamma \int_{t_\gamma}^{T_\gamma} \rho_{T_0}(\gamma(t)) \cdot |\gamma'(t)| dt \geq \inf_\gamma \int_{t_\gamma}^{T_\gamma} \frac{|\gamma'(t)|}{2|\gamma(t)|} dt \geq \frac{1}{2} |\log|w_{i+1}| - \log|w_i||,$$

where $\gamma : [t_\gamma, T_\gamma] \to T_0$ is any rectifiable curve that connects w_0 and w_1. Hence

$$|w_{i+1}| \leq \mathrm{e}^{2A} |w_i|.$$

The claim now follows with $K = \mathrm{e}^{2A} > 1$.

Recall that U contains a punctured disk at ∞, hence

$$\rho_U(z) \leq \mathrm{O}\left(\frac{1}{|z| \cdot \log|z|}\right) \quad \text{as } z \to \infty.$$

On the other hand, $V \subset \mathbb{C} \setminus \{w_n\}$, and it follows from [Rem, Proposition 2.1] that

$$\frac{1}{\rho_V(z)} \leq \mathrm{O}(|z|) \text{ as } z \to \infty.$$

Hence $\rho_U(z)/\rho_V(z) \to 0$ as $z \to \infty$ and the statement follows. $\square$

Proposition 4.6. *Assume that z_0 is an isolated boundary point of U. Then for every horosphere $H_\varepsilon(\infty)$ there exists a horosphere $H_\delta(z_0)$ such that $H_\delta(z_0) \subset U \backslash \overline{H_\varepsilon(\infty) \cup f^{-1}(H_\varepsilon(\infty))}$ and $f(H_\delta(z_0)) \supset H_\delta(z_0)$. Furthermore, δ can be replaced by any $\tilde{\delta} < \delta$.*

Proof. First recall that z_0 is necessarily a repelling fixed point of f.

The first statement is obvious since $z_0 \notin \overline{H_\varepsilon(\infty)}$ holds for any horosphere $H_\varepsilon(\infty)$ and since z_0 is a fixed point of f.

As introduced in Section 2.2, there exist a covering map $\pi : \mathbb{D}^* \to U$ and a constant $0 < \tau < 1$ such that π maps $D_\tau(0) \setminus \{0\}$ one-to-one to the horosphere $H_\tau(z_0) := \pi(D_\tau(0) \setminus \{0\})$ at z_0. By the Riemann Removable Singularity Theorem, the embedding $\pi|_{D_\tau(0)\setminus\{0\}}$ can be continued holomorphically to 0.
For any $\delta < \tau$ let $h_\delta(z_0) = \pi(S^1_\delta)$, where $S^1_\delta := \partial D_\delta(0)$, and denote by $i(\delta)$ and $o(\delta)$ its inner and outer radius, respectively. Clearly, $i(\delta), o(\delta) \to 0$ as $\delta \to 0$, as well as

$$\frac{o(\delta)}{i(\delta)} \to 1 \quad \text{as } \operatorname{dist}(h_\delta(z_0), z_0) \to 0.$$

By composing f with a linear transformation, we can assume that $z_0 = 0$, so the power series of the function f has the form

$$f(z) = \mu(0) \cdot z + O(z^2)$$

in a neighbourhood of 0, where $\mu(0)$ is the multiplier of 0. Let $|z| = i(\delta)$. Then

$$\left|\frac{f(z)}{z}\right| \geq \frac{|\mu(0)| \cdot i(\delta) - O(i(\delta)^2)}{i(\delta)} = |\mu(0)| - O(i(\delta)) > \frac{o(\delta)}{i(\delta)}$$

for every sufficiently small δ. Hence every point $z \in h_\delta(0)$ is mapped outside the circle at 0 with radius $o(\delta)$ and the statement follows. □

Recall that our goal in Section 4.1 is to prove Theorem 4.4 which states that the iteration of the leg map produces only finitely many equivalence classes of legs. Since equivalence classes of legs arise, roughly speaking, by winding around components of ∂U, we want to find a compact subset Y of U so that producing additional homotopy implies increase of length of leg-pieces contained in Y. By choosing Y so that Proposition 4.5 applies, we can later use uniform contraction arguments to control the lengths of the considered pieces of legs.
Note that if γ is *any* leg, it is fairly impossible to make useful statements about the location of its iterated images $\mathscr{L}^n(\gamma)$ related to an *arbitrary* compact set Y. Only by constructing Y carefully using hyperbolic geometry and working with geodesic legs rather than arbitrary legs, we obtain additional tools that enable us to control the lengths of geodesic leg-pieces.

Theorem 4.7. *There exists a compact path-connected set $Y \subset U$ with finitely many boundary components such that:*

(*a*) *If g is a geodesic leg, then $g \cap Y$ is non-empty and connected. Furthermore, if K_1 and K_2 are distinct components of $\widehat{\mathbb{C}} \setminus U$, then K_1 and K_2 are contained in two distinct components of $\widehat{\mathbb{C}} \setminus Y$.*

(*b*) *Let $C(z_0)$ and $C(\infty)$ denote the components of $U \setminus Y$ that contain z_0 and ∞, respectively, as boundary points, and for a leg γ, denote by $\tilde{\ell}_U(\gamma)$ the hyperbolic length in U of the longest subpiece of γ connecting the boundaries of $C(z_0)$ and $C(\infty)$ in U. Then there exists a constant $0 < P < \infty$, such that if g is a geodesic leg and $\gamma \in [g]$ is another leg, then $\tilde{\ell}_U(g) \leq \tilde{\ell}_U(\gamma) + P$.*

(*c*) *There exists a constant $0 < M < \infty$, such that if g is a geodesic leg, then there exists a leg $\gamma_1 \in [\mathscr{L}(g)]$ with $\tilde{\ell}_U(\gamma_1) \leq \ell_U(\mathscr{L}(g) \cap f^{-1}(Y)) + M$.*

Proof. Let $p_0, \dots, p_n$ be the punctures of U including ∞, and let us assume that $p_n = \infty$. For every $i = 0, \dots, n$ choose a sufficiently small horosphere $H_{\delta_i}(p_i)$ which satisfies the conclusion of Lemma 2.3, and such that $\overline{H_{\delta_i}(p_i)} \cap \overline{H_{\delta_j}(p_j)}$ whenever $i \neq j$. Recall from Section 2.5 that $f^{-1}(H_{\delta_n}(\infty))$ is a countable union of tracts T_i, and any compact subset of U can intersect only finitely many tracts. If z_0 is one of the punctures, say $z_0 = p_0$, we also require that $H_{\delta_0}(z_0)$ satisfies the conclusion of Proposition 4.6.

Define

$$Y_1 := U \setminus \bigcup_{i=0}^{n} H_{\delta_i}(p_i).$$

Case I: $\mathbb{C} \setminus U$ *is finite.*

Let $Y := Y_1$. Clearly, Y is a compact path-connected set with finitely many boundary components.

(*a*): If g is a geodesic leg, then g does not intersect $h_{\delta_i}(p_i)$ for all $1 \leq i \leq n-1$, while it intersects $h_{\delta_0}(z_0)$ and $h_{\delta_n}(\infty)$ exactly once [Hub06, Proposition 3.3.9], so in particular $g \cap Y$ is non-empty and connected. Since every puncture of U belongs to a unique component of $\widehat{\mathbb{C}} \setminus Y$, statement (*a*) follows.

(*b*): Observe that among all curves in a given homotopy class which connect the two horocycles $h_{\delta_0}(z_0)$ and $h_{\delta_n}(\infty)$, the unique geodesic realizes the smallest distance, hence the claim follows with $P = 0$.

(c): If g is any geodesic leg, then, by (a), g intersects $\partial C(z_0)$ and $\partial C(\infty)$ exactly once, while it does not enter any other horosphere. Also recall that by Proposition 4.6, the inverse branch of f that maps z_0 to itself maps $C(z_0)$ into itself. Hence the only components of $Y \setminus f^{-1}(Y)$ that might have non-empty intersection with $\mathscr{L}(g) \cap Y$ are domains that arise as the intersection of $U \setminus H_{\delta_n}(\infty)$ and a tract T (a component of $f^{-1}(H_{\delta_n}(\infty))$), that possibly contains $\mathscr{L}(g)$. Observe that $\mathscr{L}(g)$ intersects ∂T in exactly one point, say $w_0 = \mathscr{L}(g)(t_0)$, while it is possible that $\mathscr{L}(g)$ has more than one intersection point with $h_{\delta_n}(\infty)$ lying in T (see Figure 4.2). Let $w_1 := \mathscr{L}(g)(t_1)$ be the last intersection point of $\mathscr{L}(g)$ and $h_{\delta_n}(\infty)$.
If $t_0 \geq t_1$, then the longest subpiece of $\mathscr{L}(g)$ connecting $\partial C(z_0)$ and $h_{\delta_n}(\infty)$ is itself a subpiece of $\mathscr{L}(g) \cap f^{-1}(Y)$ and the claim follows with $\gamma_1 = \mathscr{L}(g)$ and $M = 0$.
Otherwise, let $\mathscr{L}'(g)$ denote the subpiece of $\mathscr{L}(g)$ connecting w_0 and w_1. Clearly, $\mathscr{L}'(g) \subset T$. Since $g|_{H_{\delta_n}(\infty)}$ is a geodesic in $H_{\delta_n}(\infty)$, it follows that $\mathscr{L}(g)|_T$ is a geodesic in T, hence $\mathscr{L}'(g)$ is a subpiece of a geodesic in T connecting w_0 to ∞. Recall that ∂T is an analytic curve, hence $\ell_U(\mathscr{L}'(g))$ depends continuously on the point w_0. Furthermore, the set of those points $w \in \partial T$ for which the geodesic in T from w to ∞ intersects $h_{\delta_n}(\infty)$ is a compact subset of U. Together with the fact that only finitely many tracts intersect the set Y, this implies that there is a finite number M such that $\ell_U(\mathscr{L}'(g)) \leq M$. Hence $\tilde{\ell}_U(\mathscr{L}(g)) \leq \ell_U(\mathscr{L}(g) \cap f^{-1}(Y)) + M$.

Case II: $\mathbb{C} \setminus U$ *is infinite.*

It follows from Definition 4.2(a), (c) that $\mathbb{C} \setminus U$ is a finite union of compact sets. Let $K_1, \ldots, K_m$ be those components of $\mathbb{C} \setminus U$ that are not punctures of U.
Now let K_i be a component such that $K_i \cap \{z_0\} = \emptyset$. By the Plane Separation Theorem [Why71, Chapter VI, Theorem 3.1] there exists a simple closed curve J_i, entirely contained in U, which separates K_i from any other component of $\widehat{\mathbb{C}} \setminus U$. By [Hub06, Proposition 3.3.8, Proposition 3.3.9], there is a unique geodesic α_i which is a simple closed curve homotopic to J_i. Denote by $\hat{\alpha}_i$ the component of $U \setminus \alpha_i$ whose boundary consists of $\alpha_i \cup \partial K_i$. Let

$$Y_2 := Y_1 \setminus \bigcup_{i=0}^{m} \hat{\alpha}_i,$$

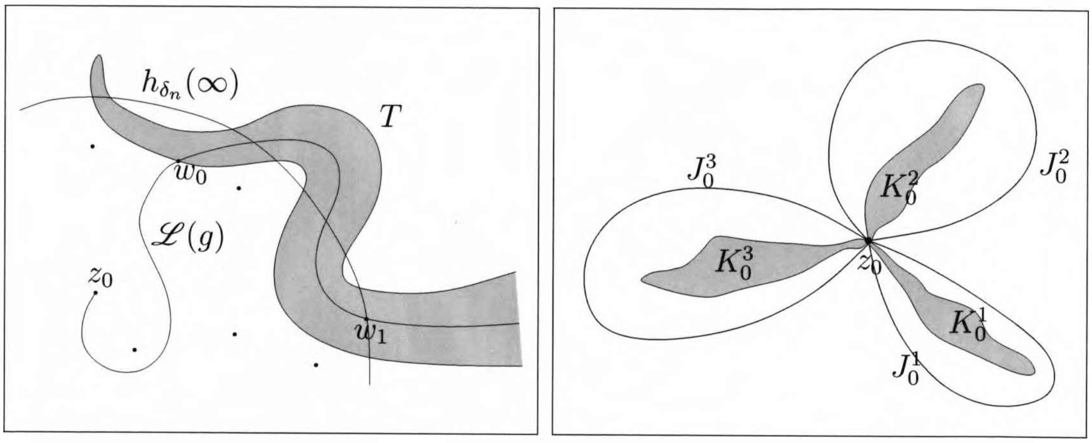

Figure 4.2: *left*: The image $\mathscr{L}(g)$ of a geodesic leg g: $\mathscr{L}(g)$ intersects the boundary ∂T of the tract T exactly once, while it can have more than one intersection point with $h_{\delta_n}(\infty)$. *right*: The point z_0 is a non-isolated boundary point contained in the component K_0 of $\mathbb{C}\backslash U$: the separating curves J_0^j intersect only in z_0.

where the union is taken over all components K_i of $\mathbb{C}\backslash U$ such that $K_i \cap \{z_0\} = \emptyset$. By [Hub06, Proposition 3.3.9] any two geodesics $\alpha_i \neq \alpha_j$ are disjoint and by our initial choice, any two horocycles $h_{\delta_j}(p_j)$ or geodesics α_i are disjoint as well. Hence the obtained set Y_2 is a subset of U with finitely many boundary components, each of which is either a horocycle $h_{\delta_j}(p_j)$ or a simple closed geodesic α_i.

Case IIa: *z_0 is a puncture of U.*

Define $Y := Y_2$. By construction, Y is a compact and path-connected set with finitely many boundary components.

(a): Let g be a geodesic leg. Since the boundary of Y consists of geodesics and horocycles and since g intersects only the horocycles at z_0 and ∞, it follows that $g \cap Y$ is connected. Furthermore, it follows from the previous construction that every component of ∂Y surrounds exactly one component of ∂U, hence (a) follows.

(b)-(c): These statements follow by exactly the same arguments as in case I.

Case IIb: *z_0 is not a puncture of U.*

Let K_0 be the component of $\mathbb{C} \setminus U$ that contains z_0. By Definition 4.2(d), $K_0 \setminus \{z_0\}$ has finitely many components, say $K_0^1, \dots, K_0^l$. It follows from the Plane Separation Theorem that for every $j = 1, \dots, l$ there is a simple closed curve $J_0^j \subset U \cup \{z_0\}$ that separates K_0^j from every component of $\widehat{\mathbb{C}} \setminus U$ other than K_0, as well as from every K_0^i, where $i \neq j$. Furthermore, $J_0^j \cap \partial U = \{z_0\}$ (see Figure 4.2). Each J_0^j is homotopic relative the start- and endpoint z_0 to a unique geodesic β^j in U, and any two such geodesics β^j and β^k intersect only in z_0. For every $j = 1, \dots, l$ let $\hat{\beta}^j$ be the component of $U \setminus (\beta^j \cup \{z_0\})$ bounded by ∂K_0^j, β^j and $\{z_0\}$ (see Figure 4.3), and let

$$Y_3 = Y_2 \setminus \bigcup_{j=1}^{l} \hat{\beta}^j.$$

It follows that $Y_3 \cap \partial U = \{z_0\}$ and that z_0 is accessible through exactly l sectors, each of which lies between two geodesics β^j and β^{j+1} (modulo l).
Let λ_j, $j = 1, \dots, l$, be a collection of simple geodesic arcs, each of which connects a point in β^j to a point in β^{j+1}, such that the domain Λ_j bounded by λ_j, $\{z_0\}$, β^j and β^{j+1} is a simply-connected subdomain of U (see Figure 4.3). Define

$$Y := Y_3 \setminus \bigcup_{j=1}^{l} \Lambda_i.$$

(a): The statement follows by exactly the same arguments as in case IIa.
(b): Let $C(z_0)$ denote the unique component of $\mathbb{C} \setminus Y$ that contains z_0 and set $P = \ell_U(\partial C(z_0)) + \ell_U(h_{\delta_n}(\infty))$. Now, $g \cap Y$ does not necessarily realize the shortest distance between $\partial C(z_0)$ and $h_{\delta_n}(\infty)$ in its homotopy class; still, if γ is a leg in $[g]$, then there exists a piece γ' of γ connecting $\partial C(z_0)$ and $h_{\delta_n}(\infty)$ which is homotopic to $g \cap Y$ relative $\partial C(z_0) \cup h_{\delta_n}(\infty)$ and we obtain $\tilde{\ell}_U(g) = \ell_U(g \cap Y) \leq \tilde{\ell}_U(\gamma') + P \leq \tilde{\ell}_U(\gamma) + P$.
(c): Recall that there is a unique $j \in \{1, \dots, l\}$ such that g intersects λ_j; their intersection point, say s, is unique and the piece of g connecting z_0 and s is entirely contained in Λ_j. Let $\tilde{\Lambda}_j$ be the component of $f^{-1}(\Lambda_j)$ such that $\mathscr{L}(g)$ intersects $\partial\tilde{\Lambda}_j$ and let $\tilde{s}$ be their unique intersection point. Observe that the piece of $\mathscr{L}(g)$ that connects z_0 and $\tilde{s}$ is entirely contained in $\tilde{\Lambda}_j$.
If $\mathscr{L}(g) \cap \tilde{\Lambda}_j \cap (U \setminus C(z_0)) = \emptyset$, then the situation is reduced to the previous case and we can choose $\gamma_1 = \mathscr{L}(g)$.
Otherwise, we replace the subpiece of $\mathscr{L}(g)$ that connects z_0 and $\tilde{s}$ by the unique homotopic geodesic ζ of the hyperbolic domain $\tilde{\Lambda}_j$ connecting z_0 and $\tilde{s}$.

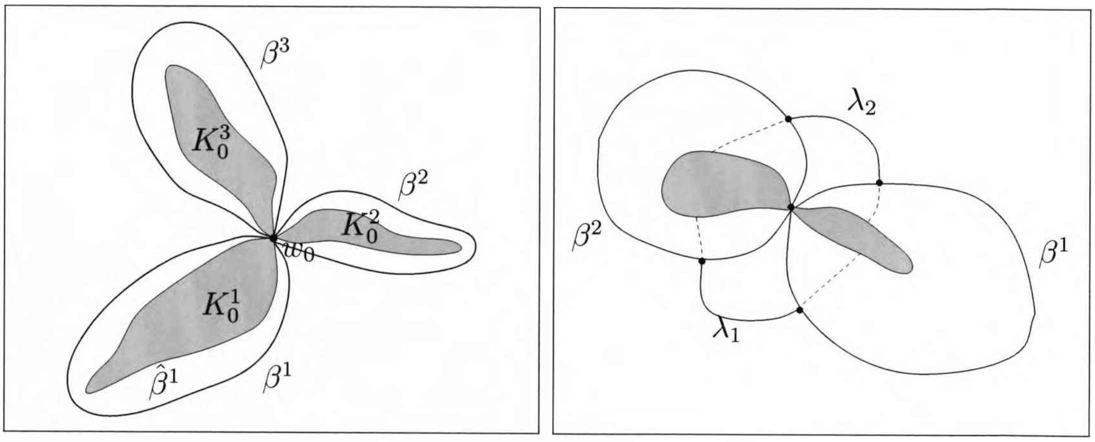

Figure 4.3: *left*: Every region $\hat{\beta}^j$ is bounded by $\beta^j \cup \partial K_0^j \cup \{z_0\}$. *right*: Geodesics λ_1 and λ_2, one for each sector through which z_0 is accessible.

If $\tilde{s} \in C(z_0)$, then we are only interested in the longest piece of ζ connecting two points in $\partial C(z_0)$, and otherwise in the longest piece of ζ that connects $\partial C(z_0)$ and $\tilde{s}$. Again, by continuity and compactness arguments (as in the case of intersections with tracts), it follows that the length in U of every such piece is globally bounded. The claim now follows from the fact that there are only finitely many domains $\tilde{\Lambda}_j$.

□

Proof of Theorem 4.4. Let Y be a compact set that satisfies the conclusions of Theorem 4.7. By Proposition 4.5 there exists a constant $\eta < 1$ such that $\rho_U(z) < \eta \cdot \rho_V(z)$ holds for all $z \in f^{-1}(Y)$. Hence if $c \subset f^{-1}(Y)$ is any rectifiable curve (on which f is injective), then $\ell_U(c) < \eta \cdot \ell_U(f(c))$.
Let g be a geodesic leg. By Theorem 4.7(c) there exists a universal constant $M > 0$ and a leg $\gamma_1 \in [\mathscr{L}(g)]$ such that $\tilde{\ell}_U(\gamma_1) \leq \ell_U(\mathscr{L}(g) \cap f^{-1}(Y)) + M$. Together with the uniform contraction this yields the estimate

$$\tilde{\ell}_U(\gamma_1) \leq \ell_U(\mathscr{L}(g) \cap f^{-1}(Y)) + M < \eta \cdot \ell_U(g \cap Y) + M.$$

Let $g_1 \in [\mathscr{L}(g)]$ be a geodesic leg. By Theorem 4.7(a), $g_1 \cap Y$ is connected, hence $\tilde{\ell}_U(g_1) = \ell_U(g_1 \cap Y)$. It then follows from Theorem 4.7(b) that there exists a universal constant $P > 0$ such that $\tilde{\ell}_U(g_1) \leq \tilde{\ell}_U(\gamma_1) + P$. Altogether, we obtain

$$\ell_U(g_1 \cap Y) = \tilde{\ell}_U(g_1) \leq \tilde{\ell}_U(\gamma_1) + P \leq \eta \cdot \ell_U(g \cap Y) + M + P.$$

By proceeding inductively it follows that if $g_n \in [\mathscr{L}^n(g)]$ is the geodesic leg, then

$$\ell_U(g_n \cap Y) < \eta^n \cdot \ell_U(g \cap Y) + (P + M) \cdot \sum_{i=0}^{n-1} \eta^i.$$

So in particular, if $L > \frac{P+M}{1-\eta}$ and $\ell_U(g \cap Y) < L$, then $\ell_U(g_n \cap Y) < L$. Recall that by Theorem 4.7 every component of $\widehat{\mathbb{C}} \setminus Y$ contains exactly one component of $\widehat{\mathbb{C}} \setminus U$, hence there can be only finitely many geodesic legs with globally bounded length. So for all $n \in \mathbb{N}$, the geodesic legs g_n belong to only finitely many equivalence classes. □

4.4 Admissible expansion domains of geometrically finite maps

We will now show that geometrically finite maps have admissible expansion domains. Such maps provide us with many examples to which our main result will apply. Still, there are functions that admit expansion domains but are not geometrically finite; an example will be given at the end of this section.

Proposition 4.8. *Let f be a geometrically finite map and let $z_0 \in \mathcal{J}(f)$ be a fixed point of f. Then f has an admissible expansion domain U at z_0. Furthermore, if f is postsingularly finite, then U can be chosen such that $\mathbb{C} \setminus U$ is finite.*

Proof. Case I: *$P(f)$ is finite.*

By Theorem 2.11, f cannot have any parabolic cycles. If $z_0 \notin P(f)$, then define $U := \mathbb{C} \setminus (P(f) \cup \{z_0\})$. Otherwise, since every function in class $\mathscr{B}$ has infinitely many repelling fixed points [LZ98, Theorem 2], there is a repelling fixed point w of f that belongs to $\mathbb{C} \setminus P(f)$. In this case we define $U := \mathbb{C} \setminus (P(f) \cup \{w\})$. In both cases there is a point in $\mathbb{C} \setminus (U \cup P(f))$ that has preimages arbitrarily close to ∞.

The set U is open, connected and finitely-connected, ∞ is an isolated boundary point and $\mathbb{C} \setminus U$ is a finite union of points. Since $f(P(f)) \subset P(f)$, it follows that $f^{-1}(U) \subset U$. But the set $f^{-1}(\mathbb{C} \setminus U)$ is not compact, hence it follows that $f^{-1}(P(f)) \neq P(f)$ and $f^{-1}(U) \subsetneq U$. Furthermore, z_0 is an isolated boundary point and hence accessible from U.

Case II: $P(f)$ *is infinite.*

Recall that in this case $\mathcal{F}(f) \neq \emptyset$. By Propositions 3.1 and 3.2 there exists a set K which can be written as a finite union of closed simply-connected domains such that $P_{\mathcal{F}} \subsetneq K \subset (\mathcal{F}(f) \cup \mathrm{Par}(f))$ and $f(K) \subsetneq K$. We define

$$U := \mathbb{C} \setminus (P(f) \cup \{z_0\} \cup K).$$

Since U is the complement of a full set, it follows that U is a domain with ∞ as an isolated boundary point. Also, $U \neq \mathbb{C} \setminus P(f)$ since $P_{\mathcal{F}} \subsetneq K$. Furthermore, $f(P(f) \cup \{z_0\} \cup K) \subsetneq (P(f) \cup \{z_0\} \cup K)$, hence $f^{-1}(U) \subsetneq U$. Since ∂K has finitely many components and since $P_{\mathcal{J}}$ is a finite set, it follows that U is finitely-connected.

If z_0 is repelling, then it is a puncture of U and in particular an accessible boundary point. If z_0 is parabolic, then it belongs to a non-trivial component K_0 of K, which is disjoint from the repulsion vectors at z_0, hence z_0 is accessible [Mil06, Lemma 10.5]. By construction (see proof of Proposition 3.2), K_0 is a union of finitely many petals at z_0, hence $K_0 \setminus \{z_0\}$ has finitely many components. □

Recall that every iterate of a geometrically finite map is again geometrically finite.

Corollary 4.9. *Let f be a geometrically finite map and let f^n be an iterate of f. Then for every repelling or parabolic fixed point z_n of f^n, there is an admissible expansion domain of f^n at z_n.*

Let us present a family of functions that are — as we will see — not geometrically finite, but for which admissible expansion domains can be constructed.

Example 4.10. The map

$$f(z) = \frac{12\pi^2}{5\pi^2 - 48}\left(\frac{(\pi^2 - 8)z + 2\pi^2}{z(4z - \pi^2)} \cos\sqrt{z} + \frac{2}{z}\right)$$

was introduced in [Ber02] as an example of a transcendental entire function that has a completely invariant Fatou component V, which contains an indirect singularity in its boundary. We will summarize some of the properties of f.
The map f has infinitely many critical values, all of which are contained in a closed interval $[0, y] \subset [0, \infty)$, and accumulate only at the asymptotic value 0. Furthermore, 0 is a parabolic fixed point of f and $(0, \infty)$ is contained in its basin of attraction V. In particular, $S(f) \cap \mathcal{F}(f)$ is not a compact set and f is not geometrically finite.
Since f maps $[0, \infty)$ into itself and since every singular value of f converges to 0, there is a compact interval $[0, \tilde{y}]$ that contains $P(f)$ and is mapped by f into itself. It follows that if z_0 is any repelling or parabolic fixed point of f, then the domain $U = \mathbb{C} \setminus (\{z_0\} \cup [0, \tilde{y}])$ is an admissible expansion domain of f at z_0. Moreover, if z_n is a repelling or parabolic fixed point of an iterate f^n, then $U = \mathbb{C} \setminus (\{z_n\} \cup [0, \tilde{y}])$ is an admissible expansion domain of f^n at z_n.
More generally, let $f_\alpha(z) := \alpha f(z)$. There exists a real number $\alpha_0 > 1$ such that for all $1 < \alpha < \alpha_0$ the map f_α has an attracting fixed point $x_\alpha > 0$ whose basin of attraction V_α contains $(0, \infty)$. Since for every such α the map f_α has a repelling fixed point at 0, it follows that $0 \in \partial V_\alpha$ and so again, f_α is not a geometrically finite map. Without remarkable differences to the previous case, we can construct admissible expansion domains for every iterate f_α^n at any of its repelling or parabolic fixed points.
We note that $f_\alpha \in R^3S$; hence for every $1 \leq \alpha < \alpha_0$ the map f_α satisfies all assumptions of Theorem 4.1.

4.5 From legs to dynamic rays

This section is devoted to the proof of our main result; together with Corollary 4.9 and the results from [RRRS] it will imply Theorem 1.1 stated in the introduction (see also Corollary 4.12). Let us recall the main statement.

Theorem 4.11. *Let f be a transcendental entire function, let z_0 be a repelling or parabolic fixed point of f and assume that there is an admissible expansion domain U of f at z_0. If for any periodic external address $\underline{s}$ there exists a periodic ray of f with address $\underline{s}$, then there is a periodic ray of f landing at z_0.*

Proof. Let us pick a horosphere $H_\delta(\infty)$ in the admissible expansion domain U. Recall that the preimage of $H_\delta(\infty)$ under f is a countable union of tracts, which we will denote by T_i. Moreover, each tract T_i can be split into fundamental domains, depending on the choice of a curve that connects $h_{\delta_n}(\infty)$ to ∞ without intersecting any of the tracts. Let us fix such a static partition $\mathscr{S}$. It is necessary to give an idea of how to define, for a leg γ, an external address which is respected by homotopies.

So let γ be any leg. Note that γ does not even need to intersect a tract. On the other hand, $\mathscr{L}(\gamma)$ is eventually contained in a tract and $\mathscr{L}^2(\gamma)$ is eventually contained in a fundamental domain. The application of this procedure to any other leg in $[\gamma]$ leads to the same fundamental domain.

Let g be a geodesic leg. By Theorem 4.4 the equivalence class $[g]$ is preperiodic. Since U is also an admissible expansion domain of every f^n, we can assume, by passing to a suitable iterate, that $[g]$ is actually fixed. We can also assume that g is eventually contained in a fundamental domain in $\mathscr{S}$, say F_0, and so are its images, since by the previous discussion this is true for all $\mathscr{L}^n(g)$ with $n \geq 2$. Hence we can assign to g the fixed external address

$$\underline{s} = \overline{F_0} = F_0F_0F_0\ldots.$$

By assumption, there exists a periodic ray $g_{\underline{s}} : (t_0, \infty) \to \mathbb{C}$ with address $\underline{s}$, hence there is a constant $\tau(g_{\underline{s}}) > t_0$ such that $g_{\underline{s}}(t) \in F_0$ for all $t \geq \tau(g_{\underline{s}})$. There is also a constant $\tau(g) > 0$ so that $g(t) \in F_0$ for all $t \geq \tau(g)$. Let $\tau := \tau(g) + \tau(g_{\underline{s}})$. We homotope g to a leg $\tilde{g} \in [g]$ by keeping $g|_{[0,\tau(g)]}$ fixed, such that $\tilde{g}(\tau) = g_{\underline{s}}(\tau)$ and $\tilde{g}(t) \in F_0$ holds for all $t \geq \tau$. Note that this is always possible since every fundamental domain is a simply-connected subset of U.

Now, the tails $g|_{[\tau,\infty)}$ and $g_{\underline{s}}|_{[\tau,\infty)}$ are both entirely contained in the same fundamental domain F_0, hence we can replace $g|_{[\tau,\infty)}$ by the ray tail $g_{\underline{s}}|_{[\tau,\infty)}$, without changing the equivalence class.

When we apply $\mathscr{L}$ to the tails of g and $g_{\underline{s}}$, then the resulting curves approach ∞ through the same fundamental domain F_0, and the same holds for the following iterates. Hence, after replacing a tail of g by a tail of the dynamic ray $g_{\underline{s}}$ and applying $\mathscr{L}$, we again obtain a leg that is eventually contained in F_0. Now, we want to show that in the limit, the iteration of $\mathscr{L}$ on such a leg yields a dynamic ray that lands at z_0.

Let $\tilde{g}(\sigma) =: x_0$ be a point on $\tilde{g}$ close to z_0. The sequence of iterated images of x_0 under the leg map (more precisely, under the corresponding inverse branch of

f) will converge to the point z_0. On the other hand, it follows from Pick's Theorem that the hyperbolic length of $\mathscr{L}^n(\tilde{g}|_{(\sigma,\tau)})$ decreases. Hence the sequence $\mathscr{L}^n(\tilde{g}(\tau))$ also converges to z_0, which means that $g_{\underline{s}}$ lands at z_0. □

Remark. If z is mapped to some fixed point z_0 of f at which some periodic ray of f lands, then z itself is the landing point of a preperiodic ray of f.

Corollary 4.12. *If $f \in R^3S$ is geometrically finite then every repelling or parabolic periodic point of f is the landing point of a periodic ray of f. In particular, every singular value in $\mathcal{J}(f)$ is the landing point of a periodic ray.*

Proof. Let z_0 be an arbitrary but fixed repelling or parabolic periodic point of f of period n. Then z_0 is a repelling or parabolic fixed point of f^n and by Corollary 4.9, f^n has an admissible expansion domain at z_0. Recall that $f \in R^3S$ implies that every iterate of f also belongs to the class R^3S.
It follows from Theorem 2.20 that for every periodic external address there is a corresponding periodic ray of f^n. Theorem 4.11 finally implies that z_0 is the landing point of a periodic ray of f^n. Since every periodic ray of f^n is also a periodic ray of f and since z_0 was an arbitrary repelling or parabolic periodic point of f, we obtain the first claim.
The second claim follows now immediately, since every singular value in $\mathcal{J}(f)$ is eventually mapped onto a repelling or parabolic cycle. □

4.6 Questions and remarks

As mentioned in the introduction, Douady's theorem and the results in [RRRS] suggest the following question.

Question 4.13. *Suppose that $f \in \mathscr{B}$ has the property $P(f) \cap I(f) = \emptyset$. Is every periodic point in $\mathcal{J}(f)$ the landing point of a dynamic ray?*

The type of functions for which we proved Theorem 4.1 seems to be a natural exhaustion of our geometric methods. Hence a (partial) answer to Question 4.13 beyond Theorem 4.1 would have to employ a different approach.
As Example 4.10 shows, admissible expansion domains can be established also for other postsingularly bounded maps that are not geometrically finite. It is also plausible that admissible expansion domains can be established for some maps with certain types of Siegel disks, but clearly not for functions for which $S_{\mathcal{J}}$ is infinite.

5 Dynamical partitions and itineraries

In Section 2.5 we introduced static partitions and external addresses for a map in class $\mathscr{B}$. This combinatorial concept has been of great use in transcendental dynamics, as many articles, including [DT86, RRRS, SZ03, RS08], show. We have also seen in our proof of Theorem 4.1 a way of applying this tool. While static partitions are mainly meaningful for the dynamics near ∞, there is another combinatorial idea that is suggestive for studying the dynamics of a map on its Julia set in terms of landing of dynamic rays: *dynamical partitions and itineraries.* The idea of itineraries comes from polynomial dynamics and we will sketch the construction for a quadratic polynomial $p(z) = z^2 + c$ with connected Julia set and a strictly preperiodic critical value c.

As already mentioned in the introduction, the escaping set of p is foliated by dynamic rays that arise as preimages of straight rays under the Böttcher map. By [Mil06, Theorem 18.11], the critical value c is the landing point of a dynamic ray g of p. The two preimage rays of g then land together at the critical point 0 and separate the plane. Hence the set $\mathbb{C}\setminus f^{-1}(g \cup \{c\})$ is the disjoint union of two simply-connected domains that we label by I^0 and I^1. Now every point $z \in \mathcal{J}(p)$ whose orbit never intersects the closure of the ray g can be assigned a sequence $I_0 I_1 I_2 \dots$ — called *itinerary* — defined by $p^n(z) \in I_n$, where $I_n \in \{I^0, I^1\}$ for all $n \geq 0$. In the same manner, we can define an itinerary for every dynamic ray of p (that is never mapped to g) since dynamic rays do not intersect. It follows in particular that rays which share their landing point must have the same itinerary; the converse statement is also not hard to see.

It suggests itself to try to transfer this concept to transcendental entire maps for which the singular values (in the Julia set) are landing points of dynamic rays. This has been performed successfully for certain maps in the exponential and cosine family (see e.g. [SZ03, Sch07a]). In this section, we will follow these ideas and construct itineraries for geometrically finite maps f for which all periodic points in the Julia set, and hence all singular values in $\mathcal{J}(f)$, are landing points of preperiodic dynamic rays. We will start with a simply-connected unbounded domain $D' \subset (\mathbb{C} \setminus S(f))$ that contains $\mathcal{J}(f)$ in its closure and whose boundary satisfies certain dynamical and topological conditions; an *itinerary domain* will be a component of $D = f^{-1}(D')$. The set of all itinerary domains corresponding to D will be called a *dynamical partition* of f (with respect to D), and an *itinerary* will then be an infinite sequence of itinerary domains. Once such a construction is provided, it is possible to prove certain results that are known for polynomials or exponential and cosine maps by following similar ideas as in these special cases. As our main result in this section, we will prove — using dynamical partitions — the remaining part of the analogy of Douady's theorem.

Theorem 5.1. *Let f be a geometrically finite map for which every periodic point in $\mathcal{J}(f)$ is the landing point of a periodic dynamic ray. Then every periodic point in $\mathcal{J}(f)$ is the landing point of at most finitely many dynamic rays, all of which are periodic.*

The results from the previous section imply that the assumptions of Theorem 5.1 are satisfied whenever f is a geometrically finite map such that for any periodic external address $\underline{s}$ there exists a periodic ray of f with address $\underline{s}$. Recall that by Theorem 2.20, periodic addresses are realized for all maps in the class R^3S.

Structure of Chapter 5

The first two parts of this chapter address the construction of suitable dynamical partitions for maps as in Theorem 5.1. In Section 5.3 we explore relations between static and dynamical partitions. This will be crucial for our proof of Theorem 5.1 which will be given in Section 5.4. Finally, we will discuss the itineraries of periodic rays that land at the same point.

Throughout Chapter 5, f will denote a transcendental entire function; in fact, from Section 5.2, the considered maps are assumed to be geometrically finite.

5.1 Attracting periodic rays

As already mentioned, we need to construct a dynamically meaningful simply-connected unbounded domain D' in $\mathbb{C} \setminus S(f)$ that contains $\mathcal{J}(f)$ in its closure; the components of its preimage will constitute a dynamical partition of f.

Recall that a geometrically finite map can have attracting basins. The set $\mathcal{A}(f)$ that consists of all points that belong to an attracting basin of f has a compact intersection with $S(f)$ and hence with $P(f)$. By removing $S(f) \cap \mathcal{A}(f)$ or $P(f) \cap \mathcal{A}(f)$ from $\mathbb{C}$ we obtain an open set that is not simply-connected (and may even be disconnected). The idea is to remove a full set K with $\mathcal{A}(f) \supset K \supset S(f)$ together with a collection of curves with known dynamical behaviour, each of which connects a component of K to infinity. One part of such a curve will be a periodic dynamic ray; the other part will be a preperiodic simple curve inside an attracting basin.

Definition 5.2. Let f be a transcendental entire function and let z_0 be an attracting periodic point of f of period n. A simple curve $\alpha : (0, \infty) \to A^*(z_0)$ is called an *attracting periodic ray* of f at z_0 (of period n) if

(*i*) $f^n(\alpha(t)) = \alpha(2t)$,

(*ii*) $\lim_{t\to\infty} \alpha(t) = z_0$,

(*iii*) $\lim_{t\to 0} \alpha(t) = w$, where $w \in \partial A^*(z_0)$ is a periodic point of f of period $d|n$.

As usual, $A^*(z_0)$ denotes the immediate attracting basin of z_0. If α is an attracting periodic ray of f at z_0 then $f(\alpha)$ is an attracting periodic ray of f at $f(z_0)$. Furthermore, if f is geometrically finite then $\lim_{t\to 0} \alpha(t) = w$ must be a repelling or parabolic periodic point of f (compare Proposition 3.4).

Proposition 5.3. *Let f be geometrically finite and let z_0 be an attracting periodic point of f. Then for any point w that belongs to the unbounded component of $A^*(z_0) \setminus P(f)$, there is an attracting periodic ray of f at z_0 that contains w.*

Proof. By passing to an iterate we can assume that z_0 is a fixed point of f. Let $\widetilde{K} := P(f) \cap A^*(z_0)$ and let K be the fill-in of $\widetilde{K}$. It follows from Proposition 3.1 that there exists a Jordan domain D and an $\varepsilon > 0$ such that $(U_\varepsilon(K) \cup f(D)) \Subset D \Subset A^*(z_0)$.

Let $w \in A^*(z_0) \setminus K$. Then w has a preimage, say $\tilde{w}$, in $A^*(z_0) \setminus K$. Now we choose a simply-connected bounded domain B with $K \cup \{w, \tilde{w}\} \subset B \Subset A^*(z_0)$ and let B' be another bounded domain in $A^*(z_0)$ with $O^+(B) \subset B'$. Since $\mathrm{Crit}(f)$ is discrete, it follows that $\mathrm{Crit}(f) \cap B'$ is finite. Furthermore, since backward orbits in $A^*(z_0)$ do not accumulate in $A^*(z_0)$, it follows that the number of points in B' that are eventually mapped to $\mathrm{Crit}(f)$ is finite. Let $B'' \subset B'$ denote the set of those points.

Let $\alpha_0 : [2^0, 2^1] \to B \setminus (K \cup B'')$ be a simple curve that satisfies $\alpha_0(1) = \tilde{w}$ and $\alpha_0(2) = f(\tilde{w}) = w$, and such that $f|\alpha_0$ is injective. We proceed inductively by defining $\alpha_n : [2^n, 2^{n+1}] \to A^*(z_0)$, $\alpha_n(2t) := f(\alpha_{n-1}(t))$ for $n \geq 1$.

Since $\alpha_0 \cap S(f) = \emptyset$, there is a unique curve α_{-1} that extends α_0, i.e., $\alpha_{-1} : [2^{-1}, 2^0] \to A^*(z_0) \setminus K$ with $f(\alpha_{-1}(t)) = \alpha_0(2t)$ and $\alpha_{-1}(2^0) = \alpha_0(2^0) = \tilde{w}$. As

before, we proceed inductively (since $\alpha_k \cap P(f) = \emptyset$ for all $k \leq -1$) and define for $n < -1$ the curve $\alpha_n : [2^n, 2^{n+1}] \to A^*(z_0) \setminus K$ to be the unique curve such that $f(\alpha_{n-1}(t)) = \alpha_n(2t)$ and $\alpha_{n-1}(2^n) = \alpha_n(2^n)$. Note that there exists some $k \in \mathbb{Z}$ such that $\alpha_n \subset A^*(z_0) \setminus D$ for all $n \leq k$.
Finally, let $\alpha : (0, \infty) \to A^*(z_0)$ be defined by $\alpha(t) := \alpha_n(t)$, where $n \in \mathbb{Z}$ is the unique number such that $2^n \leq t < 2^{n+1}$. By construction, α is a curve which satisfies $f(\alpha(t)) = \alpha(2t)$ and $\lim_{t\to\infty} \alpha(t) = z_0$. It follows from Theorem 2.18 and [Rem08, Theorem B1] that $\lim_{t\to 0} \alpha(t)$ exists (in $\mathbb{C}$) and is a fixed point of f. Hence α is an attracting periodic ray at z_0. □

Remark. If f is subhyperbolic and α is an attracting periodic ray of f then the endpoint $\lim_{t\to 0} \alpha(t)$ is a repelling periodic point of f.

5.2 Dynamical partitions of geometrically finite maps

With dynamic rays, attracting periodic rays and the tools from Section 3.1, we have all the ingredients that are necessary to construct dynamically natural partitions.

Definition and Proposition 5.4. *Let f be a geometrically finite map for which every periodic point in $\mathcal{J}(f)$ is the landing point of a periodic dynamic ray. There exists an open set $D \subset \mathbb{C}$ with the following properties:*

1. *$f(D)$ is an unbounded simply-connected subdomain of $\mathbb{C} \setminus S(f)$.*

2. *$\mathbb{C} \setminus f(D)$ has finitely many components C, each of which belongs to one of the following types:*

Type I *There exists exactly one preperiodic point $c^* \in \partial C$. Furthermore, $C \setminus \{c^*\}$ has finitely many components and exactly one of them is a dynamic ray that lands at c^*, while every other component is bounded and belongs to $\mathcal{F}(f)$. Moreover, either $\partial C \cap S(f) = \emptyset$ or $\partial C \cap S(f) = \{c^*\}$ holds.*

Type II *$C \subset \mathcal{F}(f)$ and C is eventually mapped to a component of type I. There is a point $c^* \in C$ such that $C \setminus \{c^*\}$ has two components, where exactly one of them is unbounded; this component is a curve to ∞ which is eventually mapped to an attracting periodic ray. Furthermore, $\partial C \cap S(f) = \emptyset$.*

3. *If $z \in \mathcal{J}(f) \cap \partial D$ is a preperiodic point, then $O^+(z) \subset \partial D$. Moreover, there exists $N \in \mathbb{N}$ such that*

$$W := \bigcap_{n=1}^{\infty} f^n(D) = \bigcap_{n=1}^{N} f^n(D).$$

Furthermore, $f(D) \setminus W \subset \mathcal{J}(f)$, and if a component B of ∂W separates $f(D)$ then $B \setminus \partial f(D)$ is a finite union of periodic dynamic rays landing at the same point.

Let D be a domain that satisfies the first two requirements. The set of all components of D is called a dynamical partition *of f and will usually be denoted by $\mathscr{D}(f, D)$ or $\mathscr{D}$. If D additionally satisfies the third statement then we say that $\mathscr{D}(f, D)$ is* iterative.

Remark. It is important to observe that a dynamical partition $\mathscr{D}(f, D)$ is not a subset of the plane; it is rather a set whose elements are subdomains of $D \subset \mathbb{C}$. We also could have defined $\mathscr{D}(f, D)$ to be the *union* of the components of D but we wanted to establish an analogy to static partitions as defined in Section 2.5.

Before we give a proof of Proposition 5.4, let us first state some properties of dynamical partitions that follow immediately from the definition.

Proposition 5.5. *Let f be as in Proposition 5.4 and let $\mathscr{D}(f, D)$ be a dynamical partition of f. Then $\mathcal{J}(f) \subset \overline{D}$, and every point $z \in \partial D \cap \mathcal{J}(f)$ is either on a dynamic ray or the landing point of a dynamic ray in ∂D. Furthermore, every dynamic ray of f belongs either to D or to ∂D.*

The given statements are certainly true since by Theorem 2.17, dynamic rays of f do not intersect.
Since $f(D) \subset \mathbb{C} \setminus S(f)$, the restriction $f : D \to f(D)$ is a covering map. A domain $I \in \mathscr{D}$ is called *itinerary domain.* Note that the Monodromy Theorem implies that for every itinerary domain $I \in \mathscr{D}$ the restriction $f : I \to f(D)$ is a conformal isomorphism; by the Riemann-Hurwitz Formula every I is simply-connected. Note that if a component C of $\partial f(D)$ is the union of a dynamic ray and its landing point c, then $C \setminus \{c\}$ has exactly one component.

If $z \in \mathbb{C}$ is a point such that $f^n(z) \in D$ for all $n \geq 0$ then we define the *itinerary* $\operatorname{itin}(z) = \operatorname{itin}_{\mathscr{D}}(z)$ of z to be the sequence $I_0 I_1 I_2 \ldots \in \mathscr{D}^{\mathbb{N}}$ of itinerary domains in $\mathscr{D}(f, D)$ such that $f^n(z) \in I_n$. In the same way, we can assign to every dynamic ray g, for which every iterated forward image is contained in D, a unique itinerary $\operatorname{itin}(g)$. Itineraries will usually be denoted by $\underline{i}$ or $\underline{j}$. As for external addresses, the one-sided shift map $\sigma : \mathscr{D}^{\mathbb{N}} \to \mathscr{D}^{\mathbb{N}}$ allows us to speak about itineraries that are periodic, preperiodic etc.

Note that if $\mathscr{D}(f, D)$ is an iterative dynamical partition then $f(D) \subset \mathbb{C} \setminus P(f)$. We will sometimes call a dynamical partition $\mathscr{D}(f, D)$ for which $f(D)$ has k boundary components a *dynamical k-partition.*

Proof of Proposition 5.4. By Definition 3.3 and Theorem 2.10, the set $C_A := P(f) \cap \mathcal{A}(f)$ is compact, hence there exists a finite collection $A_1, \ldots, A_n$ of components of $\mathcal{A}(f)$ such that their union covers C_A and $C_A \cap A_i \neq \emptyset$ for every i. As in the proof of Proposition 3.1, we can assume that f has only one attracting cycle, say $\{z_1, \ldots, z_m\}$, with corresponding immediate basins $A_1, \ldots, A_m$, and that $A_n \xrightarrow{f} A_{n-1} \xrightarrow{f} \ldots \xrightarrow{f} A_{m+1} \xrightarrow{f} A_m$. (Otherwise, we can repeat the argument for every other cycle.) Let $z^* \in A_n$ be a point such that for every $j \in \{0, \ldots, n-m\}$ the forward image $f^j(z^*)$ does not belong to $P(f)$, and let $C_A^* := C_A \cup O^+(z^*)$.

By Proposition 3.1, there exist finitely many Jordan domains $J_i \subset A_i$ such that their union J covers a neighbourhood of C_A^* and satisfies $f(J) \Subset J \Subset \mathcal{A}(f)$. Let $\zeta := f^{n-m}(z^*)$; then $\zeta \in J_m \setminus C_A$. By Proposition 5.3 there is an attracting periodic ray α_m at $z_m \in J_m$ that contains ζ. For $i < m$ let $\alpha_i := f^{m-i}(\alpha_m)$ denote the iterated forward images of α_m; every such curve α_i is itself an attracting periodic ray at the point z_i. Note that it is possible that $\lim_{t\to\infty} \alpha_i(t) = \lim_{t\to\infty} \alpha_k(t)$ for $i \neq k$ (see left-hand picture in Figure 5.1). Denote by α_m^* the piece of α_m that connects ζ to ∂A_m. We define α_{m+1} to be the component of $f^{-1}(\alpha_m^*)$ in A_{m+1}; we proceed recursively and define for every $j \in \{m+2, \ldots, n\}$ the curve α_j to be the component of $f^{-1}(\alpha_{j-1})$ in A_j. Note that the limit $\lim_{t\to\infty} \alpha_j(t)$ of a curve α_j defined in this way is either ∞ or a point which is mapped by f^{j-m} to the periodic point $w_m := \lim_{t\to 0} \alpha_m(t)$. Observe further that $J_j \cup \alpha_j$ is connected for every $j \in \{m+1, \ldots, n\}$ since $f^{n-j}(z^*) \in \alpha_j \cap J_j$.

For every $i \in \{1, \ldots, n\}$ define $\widetilde{K}_i := \overline{J_i \cup \alpha_i}$. By construction, every $\widetilde{K}_i$ is closed and connected and $f(\widetilde{K}_i) \subset \widetilde{K}_{i-1}$ for all $i \geq 2$ as well as $f(\widetilde{K}_1) \subset \widetilde{K}_m$.

Let $\widetilde{K}$ be the union of the sets $\widetilde{K}_i$. Then $\mathbb{C} \setminus \widetilde{K}$ is open but not necessarily connected. Nevertheless, the fill-in K of $\widetilde{K}$ is closed, connected and simply-connected. Furthermore, the Open Mapping Theorem implies that $f(K) \subset K$ (see e.g. the proof of Proposition 3.1). Hence

$$S_0 := \mathbb{C} \setminus K$$

is an unbounded domain with finitely many boundary components such that $(\mathbb{C} \setminus K) \cap (P(f) \cap \mathcal{A}(f)) = \emptyset$. Also observe that every component K_i of K that intersects the attracting cycle contains exactly one periodic point in its boundary, namely the (nonseparating) endpoint of an attracting periodic ray. Let $w_1, \ldots, w_l \in \mathbb{C}$ be the distinct points that arise as a finite limit $\lim_{t\to\infty} \alpha_i$ for some $i \in \{1, \ldots, n\}$. Recall that $l < n$ is possible. Every such limit point is eventually mapped to a periodic point in $\mathcal{J}(f)$. Let V be the minimal set that contains $\{w_1, \ldots, w_l\}$ and satisfies $f(V) \subset V$ (i.e., V is the set of forward images of the points w_i). By assumption, for every $w \in V$ there is a preperiodic dynamic ray g_w that lands at w. For every w we remove exactly one such dynamic ray from S_0; the result is the simply-connected domain

$$S_1 := S_0 \setminus \bigcup_{w \in V} g_w.$$

Note that every component of $\mathbb{C} \setminus S_1$ which contains (a preimage of) an attracting periodic ray with finite limit point, or consists of a single periodic dynamic ray, is of type I, while the other components are of type II.

Let us now consider $C_P := P(f) \cap (\mathcal{P}(f) \cup \mathrm{Par}(f))$. In a similar fashion as for C_A we can construct a finite number of closed, unbounded connected sets using (closures of) simply-connected domains from Proposition 3.2 and dynamic rays. This step is even simpler, since we do not require attracting periodic rays: by assumption, every parabolic periodic point of f is the landing point of a periodic dynamic ray. We will skip the details since they are very similar to the previous discussion. (See also the middle picture in Figure 5.1.) By removing the corresponding sets from S_1, we recover a simply-connected unbounded domain S_2 with finitely many boundary components which satisfies $f^{-1}(S_2) \subset S_2$. Also observe that $S_2 \cap P_{\mathcal{F}} = \emptyset$.

Every point in $P_{\mathcal{J}} \cap S_2$ is preperiodic, hence it is the landing point of a prepe-

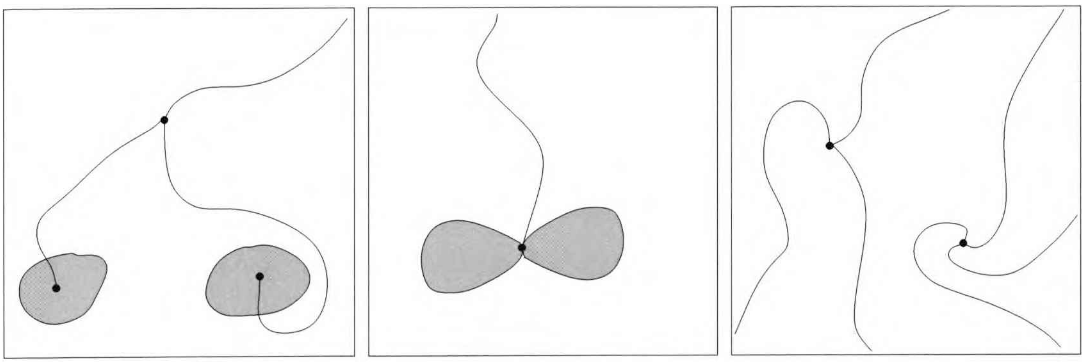

Figure 5.1: Dynamical partition - construction of the sets S_i. *left*: Two attracting periodic rays share the same endpoint at which a dynamic ray is attached. *middle*: We remove petals and a dynamic ray at a parabolic periodic point. *right*: A periodic dynamic ray can have higher period than its landing point - in this picture the landing points form a two-cycle while every dynamic ray has period 6. If we would pick more than one periodic ray at some point, then pairs of such rays sharing their landing point would separate S_3.

riodic dynamic ray. Let $p_1, \dots, p_r$ be all those points. For every i denote by $\overline{g_i}$ the union of a dynamic ray g_i (that lands at p_i) and its landing point p_i. Then

$$S_3 := S_2 \setminus \bigcup_{i=1}^{r} \overline{g_i}$$

is a simply-connected unbounded domain such that $f^{-1}(S_3) \subset S_3$ and such that the restriction $f : f^{-1}(S_3) \to S_3$ is a covering map. In fact, this holds for any iterate of f. Let $D := f^{-1}(S_3)$. By construction, the set of all components of D is a dynamical partition of f. Furthermore, it follows that the image of a point $z \in \mathbb{C} \setminus f(D)$ is contained in D only if z belongs to a periodic dynamic ray. Clearly, the period of such a ray can be higher than the period of its landing point (see right-hand picture on Figure 5.1). It is also possible that two dynamic rays $g_1, g_2 \subset \partial f(D)$ with distinct closures map to two distinct dynamic rays with the same landing point (in $\partial f(D)$). However, since the number of dynamic rays contained in $\partial f(D)$ is finite, it follows that $\mathscr{D}(f, D)$ is iterative. $\square$

Remark. Let $\mathscr{D}(f, D)$ be a dynamical partition. The proof of Proposition 5.4 yields the following bound for the number of boundary components of $f(D)$: Let F^* be the number of components of the set $\mathcal{A}(f) \cup \mathcal{P}(f) \cup \mathrm{Par}(f)$ that intersect $P(f)$ and let J^* be the number of isolated points in $P(f) \cap \mathcal{J}(f)$.

Then the proof of Proposition 5.4 implies that $f(D)$ can be chosen such that $\mathscr{D}$ is a dynamical $(F^* + J^*)$-partition.

Let $\mathscr{D}(f, D)$ be a dynamical k-partition, and let $C_1, \ldots, C_n$ be those components of $\mathbb{C} \setminus f(D)$ that contain an asymptotic value (in the boundary). Obviously, the number of these components equals $\mathcal{J}(f) \cap A(f)$. Furthermore, every such component is of type I and the asymptotic value $a_i \in C_i$ equals the unique preperiodic value $c_i^* \in C_i$. Now, for every such C_i, let us denote the number of components of $C_i \setminus \{a_i\}$ by k_i. We denote by $k_A(\mathscr{D})$ the sum of all those k_i.
If $I \in \mathscr{D}$ is an arbitrary itinerary domain, then every component of $\mathbb{C} \setminus f(D)$ that is not one of the C_i contributes exactly one preimage component to ∂I. On the other hand, a component C_i can contribute at most $k_i + 1$ preimage components. Hence we obtain the following estimate.

Corollary 5.6. *Let $\mathscr{D} = \mathscr{D}(f, D)$ be a dynamical k-partition of f. Then every itinerary domain $I \in \mathscr{D}$ has at most $k + k_A(\mathscr{D})$ boundary components.*

5.3 Relations between static and dynamical partitions

As before, let us assume in this paragraph that f is a geometrically finite map for which every periodic point in $\mathcal{J}(f)$ is the landing point of a periodic dynamic ray. For the proof of Theorem 5.1 it is necessary to discuss the following problem:
Let $\mathscr{D} = \mathscr{D}(f, D)$ be a dynamical k-partition of f and let g be a dynamic ray with a fixed itinerary $\operatorname{itin}(g) = I_0 I_0 \ldots \in \mathscr{D}^{\mathbb{N}}$. Now let $\mathscr{S} = \mathscr{S}(f, \widetilde{D}, \alpha)$ be a static partition of f. What external addresses in $\mathscr{S}^{\mathbb{N}}$ are admissible for g?

Definition 5.7. Let $I \in \mathscr{D}$ be an itinerary domain and $F \in \mathscr{S}$ a fundamental domain. We say that I *coincides* with F if $I \cap F$ is unbounded.

The next result shows that given a dynamical partition $\mathscr{D}$, the number of fundamental domains (of a suitable static partition) that coincide with a given itinerary domain $I \in \mathscr{D}$ is bounded by a constant that depends only on k.

Proposition 5.8. *Let $\mathscr{D}$ be a dynamical k-partition of f. Then there exists a static partition $\mathscr{S}$ such that every itinerary domain $I \in \mathscr{D}$ coincides with at most $k + k_A(\mathscr{D}) + 1$ fundamental domains of $\mathscr{S}$.*

Proof. Let $\mathscr{D} = \mathscr{D}(f, D)$. By assumption, $f(D)$ has k boundary components, say $C_1, \dots, C_k$, and all of them are unbounded. For every i let $c_i^* \in \partial C_i$ be a point as in Definition 5.4 (recall that if C_i is of type I then c_i^* is the unique preperiodic point in ∂C_i).

Let B be a large disk with the property that for every i, the point c_i^* and the bounded components of $C_i \setminus \{c_i^*\}$ belong to B. It follows from the type-classification in Definition 5.4 that $S(f) \subset B$, hence the components T_i of $f^{-1}(\mathbb{C} \setminus \overline{B})$ are tracts of f with respect to B.

Let C_i be a component of $\mathbb{C} \setminus f(D)$ of type I. Then $C_i \setminus B$ is a ray tail (of a dynamic ray) and hence eventually contained in a tract T_i. Let C_j be a component of type II. By Definition 5.4, $C_j \setminus B$ is a curve to ∞ which is mapped by some iterate f^n to an attracting periodic ray of f which, by construction, is contained in B. If $n = 1$ then $C_j \setminus B$ does not intersect any of the tracts $\{T_i\}$. Otherwise, $C_j \setminus B$ is eventually contained in some tract T_i. This means that we can connect ∂B to ∞ through $(\mathbb{C} \setminus \overline{B}) \cap f^{-1}(B)$ with a simple curve α such that a tail α^* of α (i.e., an unbounded subcurve of α) does not intersect $\partial f(D)$. Let $\mathscr{S} = \mathscr{S}(f, B, \alpha)$ be the static partition of f with respect to B and α. Since $\alpha^* \subset f(D)$, there is exactly one preimage component α_i^* of α^* in every itinerary domain $I \in \mathscr{D}$. On the other hand, every component of ∂I is either eventually contained in a fundamental domain in $\mathscr{S}$ or it does not intersect any of the tracts (and hence any of the fundamental domains) except in a bounded subset of the plane.

In any case, for every itinerary domain I there can be at most two (distinct) fundamental domains, say F_i and F_j, that eventually contain exactly one component of ∂I. Hence the number of fundamental domains that coincide with an itinerary domain I that has i boundary components is at most $2 + \frac{i-2}{2} = \frac{i+2}{2}$. By Corollary 5.6, $i \leq k + k_A(\mathscr{D})$, and the claim follows. □

Recall from Section 2.5 that spaces of external addresses of a given map can be identified in a natural way. Hence Proposition 5.8 immediately implies the following answer to the initial question of this paragraph.

Corollary 5.9. *Let $\mathscr{D}$ be a dynamical k-partition and let $\mathscr{S}$ be a static partition of f. Then for every $I \in \mathscr{D}$ there exist at most $m := k + k_A(\mathscr{D}) + 1$ fundamental domains $F_1(I), \dots, F_m(I) \in \mathscr{S}$ such that if $g \subset I$ is a dynamic ray then g is eventually contained in $F_i(I)$ for some $i \in \{1, \dots m\}$.*

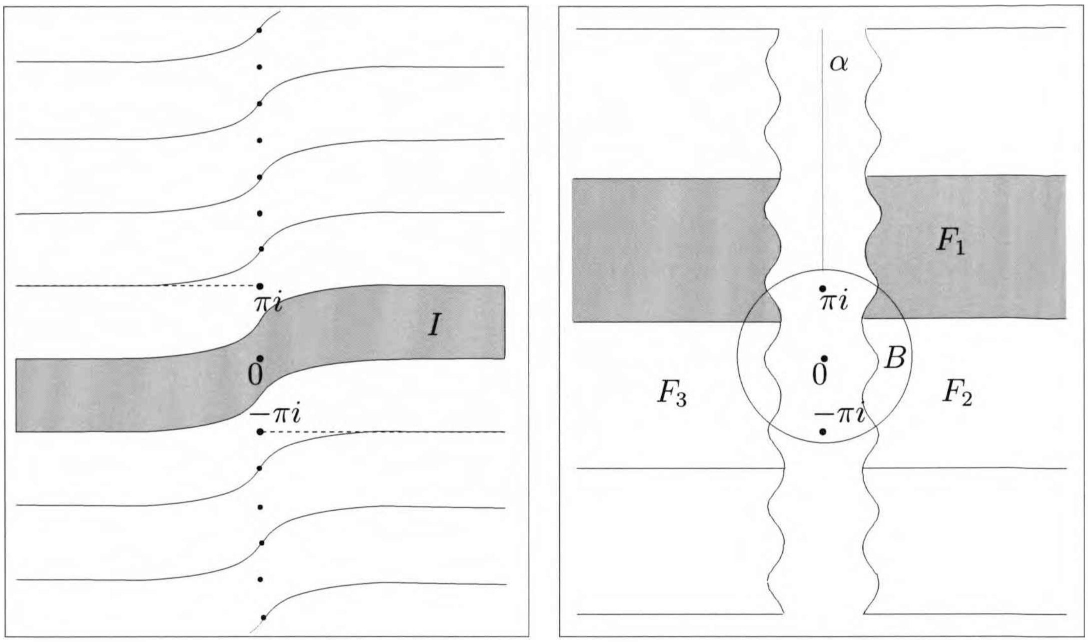

Figure 5.2: Partitions of $f : z \mapsto \pi \sinh z$. The map f has two singular values $\pm\pi i$, both of which are mapped to the repelling fixed point 0. *left*: The two horizontal lines at $\pm\pi i$ (dashed) are prefixed dynamic rays that induce the depicted dynamical 2-partition of f. *right*: The disk B contains $P(f) = \{\pm\pi i, 0\}$ and the curve α connects ∂B to ∞ without intersecting the tracts that correspond to B.
The relation between both partitions illustrates Proposition 5.8: the shadowed itinerary domain I in the left-hand picture coincides with the three fundamental domains F_1, F_2 and F_3 on the right.

5.4 Periodic points are not very hairy

Now we can prove the following, slightly more precise, version of Theorem 5.1.

Theorem 5.10. *Suppose that f is a geometrically finite map for which every periodic point in $\mathcal{J}(f)$ is the landing point of a periodic dynamic ray. Let $w \in \mathcal{J}(f)$ be a periodic point of period n. Then there exists $k \geq 1$ such that every dynamic ray that lands at w is periodic with period kn. In particular, the set of periodic rays landing at w is finite.*

Proof. Let us fix a dynamical partition $\mathscr{D} = \mathscr{D}(f, D)$ and let $\mathscr{S} = \mathscr{S}(f, B, \alpha)$ be a static partition of f. Suppose that f has a fixed ray g_0. By Theorem 2.18, g_0 has a landing point, say z_0, which is necessarily a fixed point of f. Let $\underline{s}^0 := \operatorname{addr}(g_0) = F_0 F_0 \ldots \in \mathscr{S}^{\mathbb{N}}$. Denote by R_0 the (non-empty) set of all dynamic rays that land at z_0, and by $A_0 \subset \mathscr{S}^{\mathbb{N}}$ the corresponding set of external addresses.

Since all rays in R_0 land at the same point and since they do not intersect, the set R_0 has a cyclic order which coincides with the cyclic order of R_0 at ∞. We denote this ternary relation by $C \subset R_0^3$. Note that the image of every $g \in R_0$ under f is again in R_0. Since f is a conformal map near z_0, it maps R_0 injectively into itself and the cyclic order C is preserved.

Now assume that there exists $g \in R_0$ that is not fixed. Then either $C(g_0, g, f(g))$ or $C(g_0, f(g), g)$ holds. By the properties of f, exactly one of the following cases occurs for all $n \geq 0$: $C(f^n(g), f^{n+1}(g), f^{n+2}(g))$ or $C(f^{n+2}(g), f^{n+1}(g), f^n(g))$. In particular, this means that every $h \in R_0$ is either fixed or the sequence $(f^n(h))$ is infinite.

By Proposition 5.5, z_0 either belongs to some itinerary domain in $\mathscr{D}(f, D)$ or it is the landing point of a dynamic ray that is contained in ∂D. In the latter case, let $\widetilde{C}_0$ be the component of ∂D that contains z_0 and let $C_0 = f(\widetilde{C}_0)$, i.e., C_0 is the component of $\partial f(D)$ that contains $w_0 := f(z_0)$. Since $\partial f(D)$ has only finitely many components, there is a universal constant L (depending only on $\mathscr{D}(f, D)$) such that $C_0 \setminus \{w_0\}$ has at most L components. Since z_0 is not a critical point, the number of components of $\widetilde{C}_0 \setminus \{z_0\}$ is at most L, hence there are at most L itinerary domains $I_1, \ldots, I_L$, such that $w \in \bigcap_{i=1}^L \partial I_i$. In any case, Corollary 5.9 implies that there is a subset $\mathscr{S}_f$ of $\mathscr{S}$ consisting of finitely many fundamental domains such that $A_0 \subset \mathscr{S}_f^{\mathbb{N}}$.

Let $\operatorname{addr}(g) = F_0 F_1 F_2 \ldots \in \mathscr{S}_f^{\mathbb{N}}$. Since g is not fixed, there exists a (unique) smallest integer $n_1 > 0$ such that $F_0 \neq F_{n_1}$. Since f preserves the cyclic order C, it follows for all $n \geq n_1$ that $F_n \neq F_0$. Since the orbit of g is infinite, there must be a smallest integer $n_2 > 0$ such that $F_n \neq F_{n_1}$ for all $n \geq n_2$. By repeating this argument successively, it follows that there is an integer $n_m > 0$ such that $F_k = F_{n_m}$ for all $k \geq n_m$, contradicting the assumption that the orbit of g is infinite.

Let g_0 be a periodic ray of f of period n_0. Then g_0 has a landing point, say z_0, which must be a periodic point whose period divides n_0. Since $f^{n_0}(g_0) = g_0$, it follows from the previous argument that every other dynamic ray that lands at z_0 must be periodic, and the period of every such ray must be n_0. If namely there was a dynamic ray g whose period n strictly divides n_0, then the above argument would yield that every other dynamic ray landing at z_0 would be fixed under f^n, contradicting the fact that the period n_0 of g_0 is higher than n. Hence

all dynamic rays that land at z_0 are periodic rays of the same period as g_0.
So let $w \in \mathcal{J}(f)$ be a periodic point of period n. By assumption, there is a periodic dynamic ray g that lands at w. Clearly, the period of g is a multiple of n and by the previous argument, all dynamic rays that land at w are also periodic and have the same period as g. Since two distinct dynamic rays cannot have the same external address [Rem07a, Corollary 3.4], it follows that the set of dynamic rays landing at w is finite. □

5.5 Periodic dynamic rays landing together

As before, let f be a geometrically finite map for which every periodic point in $\mathcal{J}(f)$ is the landing point of a periodic ray.
By Theorem 5.10, every periodic point $w \in \mathcal{J}(f)$ is the landing point of a finite number of dynamic rays, all of which are periodic and have the same period. Using *iterative* dynamical partitions, we can encode which periodic rays land together. Here, we will follow the same idea as in the exponential case [SZ03].
Let $\mathscr{D} = \mathscr{D}(f, D)$ be an iterative dynamical partition of f. Recall from Definition 5.4 that every periodic point of f that is in $\mathcal{J}(f)$ either has an itinerary or its entire forward orbit is contained in ∂D. Clearly, there are only finitely many periodic points whose orbit belongs to ∂D and one can determine the combinatorics at such points explicitly. Hence we will focus on those points that have a well-defined itinerary.

Proposition 5.11. *Let $\mathscr{D}(f, D)$ be an iterative dynamical partition of f and let $z \in \mathcal{J}(f)$ be a periodic point with itinerary* $\operatorname{itin}(z)$*. If w is a periodic point with* $\operatorname{itin}(w) = \operatorname{itin}(z)$ *then $w = z$.*
Let $g, \tilde{g}$ be periodic dynamic rays with itineraries and landing points in D. Then g lands at z_g if and only if $\operatorname{itin}(z_g) = \operatorname{itin}(g)$*, and g and $\tilde{g}$ land together if and only if* $\operatorname{itin}(g) = \operatorname{itin}(\tilde{g})$.

Proof. Let $\mathscr{D} = \mathscr{D}(f, D)$. As in Proposition 5.4, let $W := \bigcap_{n \geq 1} f^n(D)$; hence

$$f^{-1}(W) \subset W \cap D \subset \mathbb{C} \setminus P(f). \tag{5.1}$$

The set W is open but not necessarily connected. In any case, W has finitely many components, each of which is unbounded and has finitely many boundary components (see Definition and Proposition 5.4).

For every itinerary domain $I \in \mathscr{D}$ we define the set $W_I := W \cap I$. Thus every W_I is open and every component of W_I is a hyperbolic domain with corresponding hyperbolic metric. Observe that relation (5.1) implies that for every I there is a branch of f^{-1} that maps W into W_I. By Theorem 2.1, the restriction of every such inverse branch to any component of any W_I must be a strict contraction. Let z_1 and w_1 be two periodic points and let us assume that they have the same itinerary, i.e., $\mathrm{itin}(z_1) = \mathrm{itin}(w_1) := I_0 I_1 \ldots \in \mathscr{D}^{\mathbb{N}}$. Let $O^+(z_1) = \{z_1, z_2, \ldots, z_l\}$ and $O^+(w_1) = \{w_1, w_2, \ldots, w_m\}$, and let n be minimal such that $z_1 = z_{n+1}$ and $w_1 = w_{n+1}$. Note that we do not assume that z_1 and w_1 have the same period. Assume that there exists an integer $k \geq 1$ such that z_k and w_k belong to the same component of $W_{I_k} \subset I_k$. Then there is a branch of f^{-1} that maps z_k to z_{k-1} and w_k to w_{k-1} (modulo the respective periods); z_{k-1} and w_{k-1} again belong to the same component of $W_{I_{k-1}}$. We repeat this procedure n times and we obtain the points z_k and w_k but their hyperbolic distance has strictly decreased. This is impossible, unless $z_k = w_k$.

Let us now consider the case when for every $k \geq 1$, the iterated forward images z_k and w_k belong to different components of W_{I_k}. By Definition 5.4, for every k there is a pair of dynamic rays such that its union separates z_k and w_k. By assumption, z_{k-1} and w_{k-1} are separated in $W_{I_{k-1}}$ as well, and every such separating pair of rays must be mapped by f to a pair of rays that separates z_k and w_k. In other words, the number of separating ray pairs cannot increase under pull-backs. In particular, the fact that the orbits of z_1 and w_1 are always separated implies that there exists a periodic point $p_1 \in \partial D$ whose orbit $\{p_1, \ldots, p_r\}$ synchronizes the orbits of z_1 and w_1, i.e., $p_k \in \overline{I_k}$ for every $k \geq 1$. For simplicity, let us assume that p_1 is accessible from z_1 and w_1 within W_{I_1} (and hence the same holds for their forward images); otherwise, since W has only finitely many components, there are two points $\tilde{p}$ and $\tilde{\tilde{p}}$ in ∂D such that the forward images of the dynamic rays that land at them separate the orbits (z_k) and (w_k), and such that $\tilde{p}$ is accessible from z_1 and $\tilde{\tilde{p}}$ is accessible from w_1. Pick an integer k and a sufficiently small neighbourhood $U \subset \mathbb{C}$ of p_k such that the following holds: if p_k is repelling then $f^r(U) \supset \overline{U}$; otherwise, i.e., if p_k is parabolic, then U is the union of repelling and attracting petals (in the sense of [Mil06, Theorem 10.7]). Let $b \in \partial U \cap W_{I_k}$ and let $\alpha \subset W_{I_k}$ be a curve with finite hyperbolic length (in the metric of the respective component of W_{I_k}) that connects z_k and b. Under pull-backs of f^{lr}, b will converge to

p_k while the hyperbolic length of α will not increase. But since p_k lies in the boundary of W_{I_k}, the Euclidean lengths of the pull-backs of α must converge to 0. This implies that $z_k = p_k$. But this contradicts the assumption that z_k has an itinerary.

Now let g be a periodic dynamic ray of f. Then g has a landing point which is necessarily a periodic point. Assume that g has itinerary $\operatorname{itin}(g)$ and that the landing point z_g of g has an itinerary as well. Then clearly, $\operatorname{itin}(z_g) = \operatorname{itin}(g)$. By the first part of this proposition, the converse is true as well. In particular, it follows that two periodic rays that have an itinerary land at the same (periodic) point if and only if they have the same itinerary with respect to $\mathscr{D}$. □

We conclude with a small observation.

Corollary 5.12. *Let z be a periodic point with itinerary* $\operatorname{itin}(z)$. *Then* $\operatorname{itin}(z)$ *has the same period as z.*

Proof. First note that the period p of $\operatorname{itin}(z)$ divides the period q of z. Now assume that the claim is wrong, i.e., p strictly divides q. Then the point $w := f^p(z) \neq z$ is a periodic point with $\operatorname{itin}(w) = \operatorname{itin}(z)$, contradicting the first conclusion of Proposition 5.11. □

6 Semiconjugacies and pinched Cantor bouquets

In the present chapter we will focus on transcendental entire maps that are strongly subhyperbolic. Within this subclass of geometrically finite maps, we can strengthen our results from Chapter 4 considerably. More precisely, we will prove a theorem which allows us to describe the Julia set of *any* strongly subhyperbolic map f as a quotient of the Julia set of a disjoint type map g, where f and g can be embedded in one holomorphic one-parameter family. For simple families like $z \mapsto \lambda \sinh z$, the dynamics of disjoint type maps is well understood, and our next theorem enables us to extend this understanding to all strongly subhyperbolic maps in the corresponding family.

Theorem 6.1. *Let f be strongly subhyperbolic, and let $\lambda \in \mathbb{C}$ be such that $g(z) := f(\lambda z)$ is of disjoint type. Then there exists a continuous surjection $\phi : \mathcal{J}(g) \to \mathcal{J}(f)$, such that*

$$f(\phi(z)) = \phi(g(z))$$

for all $z \in \mathcal{J}(g)$. Moreover, ϕ restricts to a homeomorphism between the escaping sets $I(g)$ and $I(f)$.

We will see in Section 6.4 that the hypothesis of Theorem 6.1 will be automatically satisfied whenever λ is sufficiently small.
As mentioned in the introduction, the Julia set of a transcendental entire map can be the whole complex plane. But there seems to be no example of such a function for which the topological dynamics is completely understood. Theorem 6.1 enables us to provide such a description for all those maps that are strongly subhyperbolic and for which the dynamics of disjoint type maps in the same "parameter space" is well-understood. As an example, we will develop in Section 6.5 a model for the topological dynamics of the map $z \mapsto \pi \sinh z$. We choose this map since its *combinatorial* dynamics has already been extensively studied in [Sch07a, Sch07b].
For strongly subhyperbolic maps that have dynamic rays, e.g., strongly subhyperbolic maps in the class R^3S, Theorem 6.1 implies that every point in the Julia set is either on a dynamic ray or the landing point of a dynamic ray (see Corollary 6.18). For cosine maps $F_{a,b}(z) = a\,e^z + b\,e^{-z}$ with strictly preperiodic critical values, this result is due to Schleicher [Sch07a]. However, this result

alone does not explain how the escaping set of such a map is embedded in the plane. Furthermore, our proof of Theorem 6.1 formulated for subhyperbolic cosine maps can be derived in a concise and fairly elementary way, which is why the modifications for this special case are included in Section 6.3.4.
For hyperbolic maps, Theorem 6.1 is due to Rempe. In [Rem, Theorem 5.2] he constructs a sequence of conformal isomorphisms that converge to a semiconjugacy between a hyperbolic and a disjoint type map on their Julia sets. For such a limit to exist it is essential that the considered hyperbolic map f *uniformly* expands the hyperbolic metric on a domain that contains its Julia set. We transfer the rough idea to our setting. Nevertheless, the Julia set of a map f which is subhyperbolic but not hyperbolic contains singular values, hence there is no hyperbolic domain which contains $\mathcal{J}(f)$ and such that f is expanding with respect to the corresponding hyperbolic metric. We solve this problem by considering the Julia set of f as a subset of a *Riemann orbifold* $\mathcal{O}_f$ associated to f. We adopt Thurston's idea of assigning orbifolds to postcritically finite maps; this has also been used by Douady and Hubbard to prove local connectedness of Julia sets of subhyperbolic polynomials [DH84]. The main difficulty we are facing when working with metrics on Riemann orbifolds is that even for very simple hyperbolic Riemann orbifolds, there seem to be no known *global* estimates of their metrics. We will construct a hyperbolic Riemann orbifold $\mathcal{O}_f$ that contains $\mathcal{J}(f)$, for which we can compute suitable metric estimates and prove that f is uniformly expanding with respect to that metric.

Structure of Chapter 6

We start with background on Riemann orbifolds. In Section 6.2 we develop the concept of orbifolds *dynamically associated* to a strongly subhyperbolic map f; the main consequence is that we obtain a hyperbolic orbifold such that f is expanding with respect to the corresponding hyperbolic metric. It will become clear why strongly subhyperbolic maps are exactly those maps which can be approached with orbifold theory. Later, in Section 6.3, we prove that the expansion of f is actually uniform. The key for this will be Theorem 6.12 which describes the global behaviour of certain hyperbolic Riemann orbifolds. Section 6.4 addresses the construction of the semiconjugacy and the proof of several interesting corollaries. We proceed in Section 6.5 with the development of a model for the topological dynamics of the map $z \mapsto \pi \sinh z$. Finally, in Section 6.6, we will discuss some open questions on subhyperbolic transcendental entire functions.

If not stated differently, we will assume throughout Chapter 6 that the considered maps are transcendental entire.

6.1 Riemann orbifolds

An *orbifold* is a space which is locally modelled on the quotient of an open set in $\mathbb{R}^n$ by the linear action of a finite group. For a general introduction, we refer the reader to [Thu79, § 13]. We will require only orbifolds modelled on Riemann surfaces, and for a more detailed introduction to this topic see e.g. [Thu84, McM94, Mil06].

Definition 6.2 (Riemann orbifold). A *Riemann orbifold* is a pair (S, ν), where S is a Riemann surface and $\nu : S \to \mathbb{N}_{\geq 1}$ is a map called the *ramification map*, such that

$$\{z \in S \,:\, \nu(z) > 1\}$$

is discrete. A point $z \in S$ with $\nu(z) > 1$ is called a *ramified* or *marked point*. The *signature* of an orbifold is the list of values that the ramification map ν assumes at the ramified points, where a value is repeated as often as it occurs as $\nu(z)$ for some ramified $z \in S$.

Remark. The objects that we call (Riemann) orbifolds are also known under other names. We use the terminology introduced by Thurston. ([Thu79, p. 300] contains an amusing comment on how this name was obtained.)

A traditional Riemann surface can be regarded as a Riemann orbifold with ramification map $\nu \equiv 1$. In what follows, whenever we use the expression orbifold, we will always mean a Riemann orbifold.

A holomorphic map $f : \widetilde{S} \to S$ between Riemann surfaces is called a *branched covering map* if every point in S has a connected neighbourhood U such that f maps any component of $f^{-1}(U)$ onto U as a proper map. Recall that a map $f : \widetilde{V} \to V$ is called *proper* if the preimage $f^{-1}(K)$ of any compact set $K \subset V$ is a compact subset of $\widetilde{V}$.

Definition 6.3 (Holomorphic map, covering). Let $\widetilde{\mathcal{O}} = (\widetilde{S}, \tilde{\nu})$ and $\mathcal{O} = (S, \nu)$ be Riemann orbifolds. A *holomorphic map* $f : \widetilde{\mathcal{O}} \to \mathcal{O}$ is a holomorphic map $f : \widetilde{S} \to S$ between the underlying Riemann surfaces such that, for each $z \in \widetilde{S}$,

$$\nu(f(z)) \text{ divides } \deg(f, z) \cdot \tilde{\nu}(z). \tag{6.1}$$

If $f : \widetilde{S} \to S$ is a branched covering map with $\nu(f(z)) = \deg(f, z) \cdot \tilde{\nu}(z)$ for all $z \in \widetilde{S}$, then $f : \widetilde{\mathcal{O}} \to \mathcal{O}$ is an *orbifold covering map.* If additionally the surface $\widetilde{S}$ is simply-connected, then we call $\widetilde{\mathcal{O}}$ a *universal covering orbifold* of $\mathcal{O}$.

Remark. In the standard terminology, where an orbifold is defined via atlases and group actions, the definition of a holomorphic map f between two orbifolds is equivalent to a *local lifting property*; if f is a covering then every such local lift can be chosen to be an embedding. For more details, see [McM94, A2].

Note that if $f : \widetilde{\mathcal{O}} \to \mathcal{O}$ is a covering then this is not necessarily true for the map $f : \widetilde{S} \to S$ between the underlying surfaces.
Recall that by the Uniformization Theorem for Riemann surfaces, every Riemann surface has a universal cover that is conformally equivalent to either $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$. The following theorem tells us that the same is true for almost all Riemann orbifolds ([McM94, Theorem A2]).

Theorem 6.4 (Uniformization of Riemann orbifolds). *A Riemann orbifold $\mathcal{O}$ has no universal covering orbifold if and only if $\mathcal{O}$ is isomorphic to $\widehat{\mathbb{C}}$ with signature (l) or (l, k), where $l \neq k$. In all other cases the universal cover is unique up to conformal isomorphism over the surface S and hence given by either $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$.*

As usual, we will call an orbifold $\mathcal{O}$ *elliptic, parabolic* or *hyperbolic* if it is covered by $\widehat{\mathbb{C}}, \mathbb{C}$ or $\mathbb{D}$, respectively.

Remark. Let $\widetilde{\mathcal{O}}, \mathcal{O}$ be orbifolds that have a universal cover. Then a map $f : \widetilde{\mathcal{O}} \to \mathcal{O}$ is a covering if and only if it lifts to a conformal isomorphism between the universal covering spaces [Mil06, Lemma E.2].

For a connected orbifold $\mathcal{O} = (S, \nu)$, the *Euler characteristic* $\chi(\mathcal{O})$ is defined to be

$$\chi(\mathcal{O}) := \chi(S) - \sum_{z \in S} \left(1 - \frac{1}{\nu(z)}\right),$$

where $\chi(S)$ denotes the Euler characteristic of the surface S. Note that ramified points cause a reduction of $\chi(\mathcal{O})$. As for Riemann surfaces, a Riemann orbifold with negative Euler characteristic is always hyperbolic. This also implies that most orbifolds are hyperbolic. For the complete list of spherical and parabolic orbifolds, see the details of [McM94, Theorem A2].

Let C be the uniformized universal covering surface of $\mathcal{O}$ (i.e., $C \in \{\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{D}\}$) and denote by $\rho_C(z)|dz|$ its unique complete conformal metric of constant curvature 1, 0 or -1, respectively. By pushing forward this metric by a universal covering map we obtain a Riemannian metric on $\mathcal{O}$ that can be written as $\rho_{\mathcal{O}}(w)|dw|$ (in terms of a local uniformizing parameter w), and $\rho_{\mathcal{O}}(w)$ is nonzero and smooth except at the ramified points of $\mathcal{O}$. We call this metric the *orbifold metric* of $\mathcal{O}$. Observe that at a ramified point, say w_0, with ramification value m, the density has a singularity of the type $|w - w_0|^{(1-m)/m}$. More precisely, if we choose a local branched covering near 0, e.g., $z(w) = (w - w_0)^m$, then the induced metric $\rho(z(w))|dz/dw| \cdot |dw|$ is smooth and nonsingular throughout some neighbourhood of 0 in the z-plane.

Note that $\rho_{\mathcal{O}}(w)|dw|$ is again a complete metric with constant curvature 1, 0 or -1, respectively, everywhere except at the marked points (which are singularities of the curvature).

We are mainly interested in hyperbolic Riemann orbifolds. The well-known Pick Theorem for hyperbolic surfaces generalizes to hyperbolic orbifolds as well and will be of great use for us.

Theorem 6.5. *[Thu84, Proposition 17.4] A holomorphic map $f : \widetilde{\mathcal{O}} \to \mathcal{O}$ between two hyperbolic orbifolds does not increase the respective hyperbolic orbifold metrics. In fact, it is a local isometry if and only if it is a covering map; otherwise f is a strict contraction.*

In particular, if $\widetilde{\mathcal{O}}$ and $\mathcal{O}$ are two orbifolds such that $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic, then $\rho_{\widetilde{\mathcal{O}}}(z) \geq \rho_{\mathcal{O}}(z)$ for all $z \in \widetilde{\mathcal{O}}$.

6.2 Dynamically associated orbifolds

Let f be a strongly subhyperbolic map. The first step towards the proof of Theorem 6.1 is to find hyperbolic orbifolds

$$\mathcal{O}_f = (S_f, \nu_f) \quad \text{and} \quad \widetilde{\mathcal{O}}_f = (\widetilde{S}_f, \widetilde{\nu}_f)$$

such that $f : \widetilde{\mathcal{O}}_f \to \mathcal{O}_f$ is expanding with respect to the hyperbolic metric on $\mathcal{O}_f$. We will make use of the following simple observation which is analogous to what we already know from Section 2.2 about expansions with respect to conformal hyperbolic metrics.

Proposition 6.6. *Let $\mathcal{O} = (S, \nu)$ and $\widetilde{\mathcal{O}} = (\widetilde{S}, \tilde{\nu})$ be hyperbolic orbifolds with metric densities $\rho_{\mathcal{O}}$ and $\rho_{\widetilde{\mathcal{O}}}$, respectively. Let $f : \widetilde{\mathcal{O}} \to \mathcal{O}$ be a covering map and assume that the inclusion $\widetilde{\mathcal{O}} \hookrightarrow \mathcal{O}$ is holomorphic but not a covering. Then*

$$\|Df(z)\|_{\mathcal{O}} := |f'(z)| \cdot \frac{\rho_{\mathcal{O}}(f(z))}{\rho_{\mathcal{O}}(z)} > 1,$$

wherever this is defined.

Proof. By Theorem 6.5, f is a local isometry, hence $\rho_{\widetilde{\mathcal{O}}}(z) = \rho_{\mathcal{O}}(f(z)) \cdot |f'(z)|$. Since the inclusion is only holomorphic, it is a strict contraction, and hence $\rho_{\widetilde{\mathcal{O}}}(z) > \rho_{\mathcal{O}}(z)$. So altogether,

$$\rho_{\widetilde{\mathcal{O}}}(z) = \rho_{\mathcal{O}}(f(z)) \cdot |f'(z)| > \rho_{\mathcal{O}}(z),$$

implying that $\|Df(z)\|_{\mathcal{O}} > 1$. □

6.2.1 Construction of $\mathcal{O}_f$ and $\widetilde{\mathcal{O}}_f$ when f is strongly subhyperbolic

We want to associate hyperbolic orbifolds $\mathcal{O}_f$ and $\widetilde{\mathcal{O}}_f$ to a strongly subhyperbolic map f such that the assumptions of Proposition 6.6 are satisfied. Roughly, we will follow the approach of Douady and Hubbard for subhyperbolic rational maps [DH84]. We need to be able to compute sufficiently good estimates of the orbifold metrics on $\mathcal{O}_f$ and $\widetilde{\mathcal{O}}_f$. Our requirements are formalized in the following proposition.

Definition and Proposition 6.7 (Dynamically associated orbifolds). *Let f be strongly subhyperbolic. Then there exist orbifolds $\mathcal{O}_f = (S_f, \nu_f)$ and $\widetilde{\mathcal{O}}_f = (\widetilde{S}_f, \tilde{\nu}_f)$ with the following properties:*

(*a*) *The set B_f of ramified points of $\mathcal{O}_f$ is a finite set that contains $P_{\mathcal{J}}$. Furthermore, there exists a point $p \in \mathcal{O}_f \setminus S(f)$ such that $\nu_f(p) = 2 \cdot k$ for some $k \geq 1$.*

(*b*) *$\mathcal{J}(f) \subset \mathcal{O}_f$ while $P_{\mathcal{F}} \cap \mathcal{O}_f = \emptyset$.*

(c) $\mathcal{O}_f$ *is a hyperbolic orbifold containing a punctured neighbourhood of* ∞.

(d) $f : \widetilde{\mathcal{O}}_f \to \mathcal{O}_f$ *is a covering map.*

(e) *The inclusion* $\widetilde{\mathcal{O}}_f \hookrightarrow \mathcal{O}_f$ *is holomorphic but not a covering map. Furthermore, if* $S_f \neq \mathbb{C}$, *then* $\overline{\widetilde{S}_f} \subset S_f$.

We say that the pair $(\widetilde{\mathcal{O}}_f, \mathcal{O}_f)$ *of Riemann orbifolds is* dynamically associated to f, *if* $\widetilde{\mathcal{O}}_f$ *and* $\mathcal{O}_f$ *satisfy* (a)-(e).

Proof. We start with the construction of S_f. If $\mathcal{F}(f) = \emptyset$, then define $S_f := \mathbb{C}$. Otherwise, the Fatou set of f consists of attracting basins only and by Proposition 3.1, there is a finite union U of pairwise disjoint Jordan domains such that $(f(U) \cup P_{\mathcal{F}}) \Subset U \Subset \mathcal{F}(f)$. We choose a set U with this property and define

$$S_f := \mathbb{C} \backslash \overline{U}. \tag{6.2}$$

Note that S_f is connected and that $\mathcal{J}(f)$ is entirely contained in S_f.
Now assume that there is a point $p \in P_{\mathcal{J}} \setminus S(f)$, such that for every point $z \in \mathrm{Crit}(f)$ with $f^n(z) = p$ there exists $k \geq 1$ with $\deg(f^n, z) = 2 \cdot k$. Then we define the ramification value of a point $z \in S_f$ to be

$$\nu_f(z) := \mathrm{lcm}\{\deg(f^m, w), \text{ where } f^m(w) = z\}. \tag{6.3}$$

If there is no point p with such a property, then we pick a repelling fixed point $p \notin P(f)$ of f. Observe that such a point exists, since every map with a bounded set of singular values has infinitely many fixed points [LZ98, Theorem 2], and since f is subhyperbolic, only finitely many of them can be non-repelling. Since $p \in \mathcal{J}(f)$, it also belongs to S_f and we define the ramification value of every point $z \in S_f \setminus \{p\}$ to be the value defined by equation (6.3), and assign p the ramification value $\nu_f(p) = 2$. Observe that in both cases, there is a point $p \in S_f$ such that $\nu_f(p)$ is a multiple of 2.
Let $\mathcal{O}_f = (S_f, \nu_f)$. Since $\nu_f(z) > 1$ if and only if z belongs to $P_{\mathcal{J}} \cup \{p\}$, the set of ramified points of $\mathcal{O}_f$ is finite. Furthermore, no critical point $c \in S_f$ belongs to a periodic cycle, and since we have assumed that the local degree at all points in $\mathcal{J}(f)$ is globally bounded by some finite constant, the ramification

value $\nu_f(z)$ is necessarily a finite number for each $z \in S_f$. Hence $\mathcal{O}_f = (S_f, \nu_f)$ is a Riemann orbifold and statements (a) and (b) follow by construction.
Next we will prove that $\mathcal{O}_f$ is hyperbolic. Observe that it is sufficient to restrict to the case when $p \in P_{\mathcal{J}}$, since every orbifold that is holomorphically included in a hyperbolic orbifold has to be hyperbolic as well. So we consider the orbifold $\mathcal{O}_f$ with $B_f = P_{\mathcal{J}}$. We will give a proof by contradiction, so let us assume that $\mathcal{O}_f$ is not hyperbolic. Since $S_f \subset \mathbb{C}$, it follows that $\mathcal{O}_f$ must be parabolic. By assumption, $P_{\mathcal{J}} \neq \emptyset$, so it follows from [McM94, Theorem A4] that $\mathcal{O}_f$ must be isomorphic to $\mathbb{C}$ with signature either (n) or $(2,2)$. This implies that $\mathcal{F}(f) = \emptyset$, hence all singular values of f belong to $\mathcal{J}(f)$.
By (a), $P_{\mathcal{J}} \subset B_f$. By Lemma 2.4, $P(f)$ contains at least two points, hence $\mathcal{O}_f$ must be isomorphic to $\mathbb{C}$ with signature $(2,2)$. Also, f has more than one singular value (see proof of Lemma 2.4). Hence f has exactly two singular values v_1 and v_2, both of which are critical values. By signature, any of their preimages is either a critical point of local degree two or a regular point. Signature $(2,2)$ also implies that $P(f) = S(f)$, meaning that both critical values are either fixed points of f or they form a two-cycle. Since $\mathcal{F}(f) = \emptyset$, v_1 and v_2 are both repelling. In particular, $\deg(f, v_1) = \deg(f, v_2) = 1$. Let $\mathcal{O}_f' = (S_f', \nu_f')$ be the orbifold which has exactly the regular preimages of v_1 and v_2 as ramified points, assigning them the ramification value two. Clearly, $\nu_f'(v_1) = \nu_f'(v_2) = 2$. Then $f : \mathcal{O}_f' \to \mathcal{O}_f$ is a covering map and since $\mathcal{O}_f$ is parabolic, so is $\mathcal{O}_f'$, which means that v_1 and v_2 are the only ramified points in $\mathcal{O}_f'$. Hence $\mathcal{O}_f' = \mathcal{O}_f$. By conformal conjugacy we can assume that $v_1 = 1$ and $v_2 = -1$. Then the map $\mathbb{C} \to \mathcal{O}_f$, $z \mapsto \cos(z)$ is a universal covering map. Since $f : \mathcal{O}_f \to \mathcal{O}_f$ is a covering map, it lifts to a conformal $\mathbb{C}$-isomorphism $g(z) = az + b$, $a \neq 0$, yielding the relation

$$f(\cos(z)) = \cos(az + b).$$

By periodicity and symmetry of the cosine map, it follows that $a \in \mathbb{Z} \setminus \{0\}$ and $b \in \pi\mathbb{Z}$. But this means that f or $-f$ is a Chebyshev polynomial, contradicting the fact that f is transcendental. Hence $\mathcal{O}_f$ is hyperbolic.
By construction, S_f is the complement of a, possibly empty, compact set, hence

(c) follows. Define

$$\widetilde{S}_f := f^{-1}(S_f)$$

and

$$\tilde{\nu}_f(z) : \widetilde{S}_f \to \mathbb{N}, \quad z \mapsto \frac{\nu_f(f(z))}{\deg(f,z)}.$$

By equation (6.3), $\tilde{\nu}_f(z)$ is a positive integer for every $z \in \widetilde{S}_f$ and by the Identity Theorem, the set of points $z \in \widetilde{S}_f$ with $\tilde{\nu}_f(z) > 1$ is discrete. Hence $\widetilde{\mathcal{O}}_f = (\widetilde{S}_f, \tilde{\nu}_f)$ is a Riemann orbifold.

Since $A(f) \cap S_f = \emptyset$, the map $f : \widetilde{S}_f \longrightarrow S_f$ is a branched covering. Furthermore,

$$\deg(f,z) \cdot \tilde{\nu}_f(z) = \deg(f,z) \cdot \frac{\nu_f(f(z))}{\deg(f,z)} = \nu_f(f(z)),$$

hence $f : \widetilde{\mathcal{O}}_f \longrightarrow \mathcal{O}_f$ is an orbifold covering map, proving statement (d).

We will show now that the inclusion $\widetilde{\mathcal{O}}_f \hookrightarrow \mathcal{O}_f$ is holomorphic but not a covering. First note that $\widetilde{S}_f \subset S_f$ by construction of S_f. Moreover, if $S_f \neq \mathbb{C}$, then $\widetilde{S}_f$ is a relatively compact subset of S_f (see equation (6.2)). Recall that by (a), there is a point $p \in \mathcal{O}_f \setminus S(f)$ such that $\nu_f(p)$ is a multiple of 2. The fact that $p \notin S(f)$ implies that p has infinitely many preimages p_i under f, and for every such preimage point we have $\deg(f, p_i) = 1$. Moreover, $\nu_f(p_i) = 1$ holds for all but finitely many of the preimages of p, since by (a), $\mathcal{O}_f$ has only finitely many ramified points. On the other hand, $\tilde{\nu}_f(p_i) = 2$, which means that $\tilde{\nu}_f(p_i) = 2 > \nu_f(p_i) = 1$, hence the inclusion is not a covering map.

Let $z \in \widetilde{S}_f$. Observe that the definition of ν_f (see equation (6.3)) together with the fact that for any point $\omega \in \mathbb{C}$ the local degree of an iterate f^m of f is given by $\deg(f^m, \omega) = \deg(f, \omega) \cdot \deg(f, f(\omega)) \cdot \ldots \cdot \deg(f, f^{m-1}(\omega))$ implies that the product $\nu_f(z) \cdot \deg(f,z)$ divides $\nu_f(f(z))$. Since $f : \widetilde{\mathcal{O}}_f \longrightarrow \mathcal{O}_f$ is a covering, $\tilde{\nu}_f(z) = \nu_f(z) \cdot \deg(f,z) = \nu_f(f(z))$. Hence $\nu_f(z)$ divides $\tilde{\nu}_f(z)$ and this proves that the inclusion $\widetilde{\mathcal{O}}_f \hookrightarrow \mathcal{O}_f$ is a holomorphic map. □

Remark. Let $(\widetilde{\mathcal{O}}_f, \mathcal{O}_f)$ be dynamically associated to f. Note that $\widetilde{\mathcal{O}}_f$ is usually not connected. However, If f has finitely many tracts over ∞, e.g., if f has finite order, then the number of components of $\widetilde{\mathcal{O}}_f$ is finite. Observe also that it follows from Proposition 6.7(d),(e) that the set B_f of ramified points of $\mathcal{O}_f$

satisfies $f(B_f) \subset B_f$.

Finally we would like to mention that we have proved more than the existence of a hyperbolic orbifold $\mathcal{O}_f$ satisfying the remaining assumptions of Proposition 6.7. In fact, we have also shown the following: *Let f be a transcendental entire function and let $\widetilde{\mathcal{O}}$ and $\mathcal{O}$ be any two orbifolds such that $f : \widetilde{\mathcal{O}} \to \mathcal{O}$ is a covering map. Then $\mathcal{O}$ (and hence $\widetilde{\mathcal{O}}$) is hyperbolic.*

Corollary 6.8. *Let f and $\widetilde{\mathcal{O}}_f$ be as in Proposition 6.7. Then there is a constant $K > 1$ and an infinite sequence of points z_i for which $\tilde{\nu}_f(z_i)$ is a multiple of 2, such that $|z_i| < |z_{i+1}| \leq K|z_i|$ holds for all i.*

Proof. Let $p \in \mathcal{O}_f \setminus S(f)$ be a point such that $\nu_f(p)$ is a multiple of 2 and let γ be a Jordan curve in $\mathbb{C}$ such that the bounded component of $\mathbb{C}\setminus\gamma$ contains $S(f)$ but not p. The components of the preimage of the unbounded component of $\mathbb{C} \setminus \gamma$ are tracts of f and every such tract contains infinitely many preimages of p. Let z_i denote the preimages of p lying in one such (arbitrary but fixed) tract. It follows from the claim in Proposition 4.5 that there is a constant $K > 1$ such that $|z_i| < |z_{i+1}| \leq K|z_i|$ holds for infinitely many i. However, since all but finitely many z_i are regular points of f, it follows that $\tilde{\nu}_f(z_i) = \nu_f(p)$ and this is, by assumption, a multiple of 2. □

Notations. For a pair $(\widetilde{\mathcal{O}}_f, \mathcal{O}_f)$ of orbifolds dynamically associated to f, we denote by $\tilde{\rho}_f$ and ρ_f the densities of the hyperbolic metrics of $\widetilde{\mathcal{O}}_f$ and $\mathcal{O}_f$, respectively.

We conclude this section with the following simple observation which justifies our restriction to strongly subhyperbolic maps.

Proposition 6.9. *Let f be a subhyperbolic map for which there is a pair of dynamically associated orbifolds. Then f is strongly subhyperbolic.*

Proof. If a is an asymptotic value of f, then for any compact set $K \subset \mathbb{C}$ containing a there exists a component of $f^{-1}(K)$ which is not compact. Hence there is no domain $U \ni a$ such that $f : f^{-1}(U) \to U$ is a proper map. Hence, if an asymptotic value of f belongs to $\mathcal{J}(f)$, then there is no domain $U \supset \mathcal{J}(f)$ such that $f : f^{-1}(U) \to U$ is a (branched) covering map.

Assume now that f has a critical value $w \in \mathcal{J}(f)$, such that for every $n \in \mathbb{N}$ there exists a point z_n with $f(z_n) = w$ and $\deg(f, z_n) \geq n$. If there was a pair

$(\widetilde{\mathcal{O}}_f, \mathcal{O}_f)$ of dynamically associated orbifolds, then Proposition 6.7(d) would imply that w is a puncture of $\mathcal{O}_f$, contradicting the fact that $\mathcal{J}(f) \subset S_f$.

□

6.3 Uniform expansion

Let f be a strongly subhyperbolic map and let $(\widetilde{\mathcal{O}}_f, \mathcal{O}_f)$ be dynamically associated to f. By Proposition 6.6,

$$\|Df(z)\|_{\mathcal{O}_f} = |f'(z)| \cdot \frac{\rho_f(f(z))}{\rho_f(z)} > 1$$

wherever defined, so in particular for all $z \in \widetilde{\mathcal{O}}_f$.

Remark. If $w \in \widetilde{\mathcal{O}}_f$ is a point such that $\nu_f(f(w)) > 1$, then it follows by Proposition 6.7(d),(e) that $\nu_f(w) \cdot \deg(f, w)$ divides $\nu_f(f(w))$. In this case we define

$$\|Df(w)\|_{\mathcal{O}_f} := \lim_{z \to w} \|Df(z)\|_{\mathcal{O}_f},$$

and so if $\nu_f(w) \cdot \deg(f, w) = \nu_f(f(w))$, then $\|Df(w)\|_{\mathcal{O}_f}$ is a finite number, while otherwise $\|Df(w)\|_{\mathcal{O}_f} = \infty$.

Our goal is to show that the expansion of f is uniform.

Theorem 6.10 (Uniform expansion). *Let f be strongly subhyperbolic and let $(\widetilde{\mathcal{O}}_f, \mathcal{O}_f)$ be dynamically associated to f. Then there is a constant $E > 1$ such that*

$$\|Df(z)\|_{\mathcal{O}_f} \geq E$$

for all $z \in \widetilde{\mathcal{O}}_f$.

The remainder of Section 6.3 is devoted to the proof of Theorem 6.10.

6.3.1 Continuity of orbifold metrics

We want to show the following continuity statement: Let $\mathcal{O}$ be a Riemann orbifold, let p be a regular and q a ramified point of $\mathcal{O}$. Then the value of the density map $\rho_{\mathcal{O}}$ of the orbifold metric at p depends continuously on q, i.e., if

we perturb the point q slightly, then the density of the corresponding orbifold metric at the point p will also undergo only a small change.
This statement is surely not new but we were not able to locate a reference. Hence we include a proof for completeness. We will restrict to orbifolds whose underlying surface is a subset of the sphere; the proof in the general case, where the underlying surface of $\mathcal{O}$ is an arbitrary Riemann surface, can be derived using exactly the same arguments, with the additional step of taking charts.
So let $S \subset \widehat{\mathbb{C}}$ and let $\mathcal{O} = (S, \nu)$. Denote by B the set of ramified points of $\mathcal{O}$. Let $n > 1$ be an arbitrary but fixed integer. For a point $q \in S \backslash B$ we define a new orbifold $\mathcal{O}_q = (S, \nu_q)$, where

$$\nu_q(z) = \begin{cases} \nu(z) & \text{if } z \neq q, \\ n & \text{if } z = q. \end{cases}$$

Furthermore, we assume that every such orbifold has a universal cover (hence we exclude the case when S is a sphere and the set B is either empty or consists of only one point with ramification value $m \neq n$). Note that any two such orbifolds $\mathcal{O}_q$ and $\mathcal{O}_{\tilde{q}}$ have the same signature and hence the same uniformized universal cover. Let us denote the density of the orbifold metric on $\mathcal{O}_q$ by ρ_q.
For a point $p \in S \setminus B$ we define the map

$$M_p : S \setminus (B \cup \{p\}) \to (0, \infty], \quad q \mapsto \rho_q(p).$$

Theorem 6.11 (Continuity of orbifold metrics). *Let $p \in S \setminus B$ be arbitrary but fixed. Then the map M_p is continuous at every point in $S \setminus (B \cup \{p\})$.*

Proof. Let $q^* \in S \setminus (B \cup \{p\})$ be an arbitrary but fixed point. We want to show that M_p is continuous at q^*.
Pick a sufficiently small Jordan domain $D \ni q^*$ such that $\overline{D} \cap (B \cup \{p\}) = \emptyset$. Let a be a point in $S \setminus (\overline{D} \cup B \cup \{p\})$. By conformal conjugacy, we can assume that $a = 0$.
For two points $z_1, z_2 \in D$ let us denote by $d_D(z_1, z_2)$ the distance between z_1 and z_2 measured in the hyperbolic metric of D. For a point $q \in D$ consider the unique Riemann map $H_q : D \to \mathbb{H}$ which maps $q^* \mapsto i$ and $q \mapsto h_q i$, where $h_q := \mathrm{e}^{d_D(q^*, q)}$. Let $L_q : \mathbb{H} \to \mathbb{H}$, $(x + iy) \mapsto x + h_q y i$. Then L_q is a

h_q-quasiconformal self-map of $\mathbb{H}$. Define

$$\varphi_q : D \to D, \quad z \mapsto H_q \circ L_q \circ H_q^{-1}(z).$$

It is easy to see that φ_q extends continuously to the complement of D as the identity map and that the extended map, which we will also denote by φ_q, is a h_q-quasiconformal map (see e.g. [GS98, Lemma 5.2.3]). Observe that $\varphi_q \to \varphi_{q^*} \equiv \mathrm{id}$ as $q \to q^*$.
Let C be the uniformized universal covering surface of $\mathcal{O}_{q^*}$ and $\mathcal{O}_q$ and let $\pi_{q^*} : C \to \mathcal{O}_{q^*}$ and $\pi_q : C \to \mathcal{O}_q$ be universal covering maps, both normalized such that $\pi_{q^*}(0) = \pi_q(0) = 0$ and $\pi_{q^*}'(0) = \pi_q'(0)$. Considered as a map between orbifolds, $\varphi_q : \mathcal{O}_{q^*} \to \mathcal{O}_q$ is a homeomorphism and hence can be lifted to a homeomorphism on C. From now on we will assume that $C = \mathbb{D}$ since the other two cases follow by the same strategy, using even simpler calculations.

Claim. There is a unique lift $\tilde{\varphi}_q : \mathbb{D} \to \mathbb{D}$ of φ_q such that $\tilde{\varphi}_q(0) = 0$.

Proof of claim. Let G_q denote the covering group of $\mathbb{D}$ over $\mathcal{O}_q$ and assume that there exist two distinct lifts $\tilde{\varphi}_q$, $\tilde{\tilde{\varphi}}_q$ of φ_q that fix 0. There exists a mapping $h \in G_q$ such that $\tilde{\tilde{\varphi}}_q(z) = h(\tilde{\varphi}_q(z))$ holds for all $z \in \mathbb{D}$. It follows from our assumption that $h(0) = 0$, hence $h \in \mathrm{Stab}(0) \subset G_q$, where Stab(0) denotes the stabilizer of 0 in G_q. But $\pi_q(0) = 0$ and 0 is a non-ramified point of $\mathcal{O}_q$, which means that $\mathrm{Stab}(0) \subset G_q$ is trivial. Hence $h \equiv \mathrm{id}$ and $\tilde{\varphi}_q \equiv \tilde{\tilde{\varphi}}_q$.
We have the following commutative diagram:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\tilde{\varphi}_q"] \arrow[d, "\pi_{q^*}"'] & \mathbb{D} \arrow[d, "\pi_q"] \\ \mathcal{O}_{q^*} \arrow[r, "\varphi_q"'] & \mathcal{O}_q \end{tikzcd}$$

Since φ_q is a h_q-quasiconformal map and π_{q^*} and π_q are holomorphic, the map $\tilde{\varphi}_q : \mathbb{D} \to \mathbb{D}$ is also h_q-quasiconformal. Moreover, $\tilde{\varphi}_q$ is conformal when restricted to the set $\Omega := \mathbb{D} \setminus \pi_{q^*}^{-1}(D)$, so in particular in a sufficiently small neighbourhood of any point in the set $\{\pi_{q^*}^{-1}(p)\}$.
Furthermore, the lifts $\tilde{\varphi}_q$ converge to $\tilde{\varphi}_{q^*}$ (locally uniformly) as $q \to q^*$ and, due

to the chosen normalization, $\tilde{\varphi}_{q^*} \equiv \text{id}|_{\mathbb{D}}$ [Hub06, Chapter 4.7]. Moreover, when restricted to Ω, the maps converge in the C^1-norm, hence $(\tilde{\varphi}_q)'|_\Omega \to (\tilde{\varphi}_{q^*})'|_\Omega$ when $q \to q^*$.

By the above diagram we can write $\pi_q(z) = (\varphi_q \circ \pi_{q^*} \circ \tilde{\varphi}_q^{-1})(z)$ for every $z \in \mathbb{D}$. Recall that if $w \in S$ and $z_q \in \{\pi_q^{-1}(w)\}$, then the value of the density function ρ_q at w is given by $\rho_q(w) = \rho_{\mathbb{D}}(z_q) \cdot (\pi_q'(z_q))^{-1}$ and this does not depend on the choice of the preimage of w. Similarly, if $z_{q^*} \in \{\pi_{q^*}^{-1}(w)\}$, then $\rho_{q^*}(w) = \rho_{\mathbb{D}}(z_{q^*}) \cdot (\pi_{q^*}'(z_{q^*}))^{-1}$. Hence,

$$|\rho_{q^*}(w) - \rho_q(w)| = \left| \frac{\rho_{\mathbb{D}}(z_{q^*})}{\pi_{q^*}'(z_{q^*})} - \frac{\rho_{\mathbb{D}}(z_q)}{\pi_q'(z_q)} \right|.$$

Observe first that $\varphi_q(p) = p$, since $p \in S \setminus D$. Let us fix a point $p_q \in \{\pi_q^{-1}(p)\}$. Then

$$p = \pi_q(p_q) = (\varphi_q \circ \pi \circ \tilde{\varphi}_q^{-1})(p_q) = (\pi \circ \tilde{\varphi}_q^{-1})(p_q). \tag{6.4}$$

Let $p_{q^*} \in \{\pi_{q^*}^{-1}(p)\}$ be the unique point such that $p_q = \tilde{\varphi}_q(p_{q^*})$. We obtain

$$\begin{aligned} \pi_q'(p_q) &= \varphi_q'((\pi_{q^*} \circ \tilde{\varphi}_q^{-1})(p_q)) \cdot \pi_{q^*}'(\tilde{\varphi}_q^{-1}(p_q)) \cdot (\tilde{\varphi}_q^{-1})'(p_q) \\ &\underbrace{=}_{(6.4)} \varphi_q'(p) \cdot \pi_{q^*}'(p_{q^*}) \cdot (\tilde{\varphi}_q^{-1})'(p_q) \underbrace{=}_{\varphi_q'(p)=1} \pi_{q^*}'(p_{q^*}) \cdot (\tilde{\varphi}_q^{-1})'(p_q). \end{aligned}$$

Hence

$$\begin{aligned} |\rho_{q^*}(p) - \rho_q(p)| &= \frac{1}{|\pi_{q^*}'(p_{q^*})|} \cdot \left| \rho_{\mathbb{D}}(p_{q^*}) - \frac{\rho_{\mathbb{D}}(\tilde{\varphi}_q(p_{q^*}))}{(\tilde{\varphi}_q^{-1})'(p_q)} \right| \\ &= \frac{1}{|\pi_{q^*}'(p_{q^*})|} \cdot \left| \frac{1}{1 - |p_{q^*}|^2} - \frac{1}{(\tilde{\varphi}_q^{-1})'(p_q) \cdot (1 - |\tilde{\varphi}_q(p_{q^*})|^2)} \right|. \end{aligned}$$

Since $\tilde{\varphi}_q \to \text{id}$ in the C^1-norm in a neighbourhood of p_{q^*} when $q \to q^*$, the expression $(\tilde{\varphi}_q^{-1})'(p_q) \cdot (1 - |\tilde{\varphi}_q(p_{q^*})|^2)$ tends to $1 - |p_{q^*}|^2$ and hence $|\rho_{q^*}(p) - \rho_q(p)| \to 0$. □

6.3.2 Estimates of metrics with infinitely many singularities

We are now able to prove the key-statement for the proof of Theorem 6.1.

Theorem 6.12. *Let $K > 1$ and let z_i, $i \in \mathbb{N}$, be an infinite sequence of points* *satisfying* $|z_i| < |z_{i+1}| \leq K|z_i|$. *Let* $\mathcal{O} = (\mathbb{C}, \nu_{\mathcal{O}})$, *where*

$$\nu_{\mathcal{O}}(z) = \begin{cases} 2 & \text{if } z = z_i \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$$

Then the density $\rho_{\mathcal{O}}$ *of the hyperbolic metric on* $\mathcal{O}$ *satisfies*

$$\frac{1}{\rho_{\mathcal{O}}(z)} \leq \mathrm{O}(|z|) \quad \text{as} \quad z \to \infty.$$

Proof. First observe that, by affine conjugacy, we can assume that 0 is one of the ramified points of $\mathcal{O}$.

Let $z \neq z_i$ be an arbitrary but fixed point in $\mathcal{O}$. Depending on z, we choose $b = b(z) = z_k$, where z_k satisfies $|z_k| \geq (K+1)|z|$ and is minimal with this property, i.e., if $|z_j| < |z_k|$ then $|z_j| < (K+1)|z|$. It follows immediately that

$$(K+1)|z| \leq |b| = |z_k| \leq K|z_{k-1}| \leq (K+1)K|z|. \tag{6.5}$$

Next we set $c = c(z) = z_l$, where z_l is minimal with the property $|z_l| \geq (K+1)|b|$. We then obtain

$$(K+1)^2|z| \leq (K+1)|b| \leq |c| \leq K|z_{l-1}| \leq (K+1)K|b| \leq (K+1)^2K^2|z|. \tag{6.6}$$

For any three pairwise distinct points $p, q, r \in \mathbb{C}$ denote by $\mathcal{O}_{p,q,r} := (\mathbb{C}, \nu_{p,q,r})$ the orbifold defined by

$$\nu_{p,q,r}(w) = \begin{cases} 2 & \text{if } w \in \{p, q, r\}, \\ 1 & \text{otherwise.} \end{cases}$$

Note that every such orbifold is hyperbolic, since its Euler characteristic equals $-1/2$. We denote by $\rho_{p,q,r}$ the density of the hyperbolic metric on $\mathcal{O}_{p,q,r}$.

Observe first that $\mathcal{O}$ is holomorphically embedded in $\mathcal{O}_{0,b,c}$, and it follows from Theorem 6.5 that $\rho_{\mathcal{O}}(w) > \rho_{0,b,c}(w)$ holds for all $w \in \mathcal{O}$. Let $\tilde{b} = \tilde{b}(z) := b/|z|$ and $\tilde{c} = \tilde{c}(z) := c/|z|$. Then the map

$$S_z : \mathcal{O}_{0,b,c} \to \mathcal{O}_{0,\tilde{b},\tilde{c}}, \quad w \mapsto \frac{w}{|z|}$$

is obviously a conformal isomorphism and hence a local isometry. Altogether, we obtain

$$\rho_{0,\tilde{b},\tilde{c}}(S(w)) = \rho_{0,b,c}(w) \cdot |w| < \rho_{O}(w) \cdot |w|. \tag{6.7}$$

Let $\tilde{z} := S(z) = z/|z|$. Then equations (6.5) and (6.6) yield

$$(K+1) \leq |\tilde{b}| \leq (K+1)K < (K+1)^2 \leq |\tilde{c}| \leq (K+1)^2K^2,$$

i.e., $\tilde{b} \in A_1 = A_1(K) := \{w : (K+1) \leq |w| \leq (K+1)K\}$ and $\tilde{c} \in A_2 = A_2(K) := \{w : (K+1)^2 \leq |w| \leq (K+1)^2K^2\}$ belong to compact nonintersecting annuli, both disjoint from $\tilde{z}$ (see Figure 6.1).

By Theorem 6.11, the map

$$D^2 : \mathbb{C} \setminus \{\tilde{z}\} \times \mathbb{C} \setminus \{\tilde{z}\} \to (0, \infty), \quad (x, y) \mapsto \rho_{0,x,y}(\tilde{z})$$

is a composition of two continuous maps and hence itself continuous. Furthermore, it attains its minimum (and maximum) on the compact set $A_1 \times A_2$. Hence, there exist constants $0 < m(K), M(K) < \infty$ depending only on K (but not on z) such that

$$m(K) < \rho_{0,\tilde{b},\tilde{c}}(\tilde{z}) < M(K).$$

By setting $w = z$ in equation (6.7), we finally get

$$m(K) \cdot \frac{1}{|z|} \leq \frac{\rho_{0,\tilde{b},\tilde{c}}(\tilde{z})}{|z|} < \rho_{O}(z),$$

and the assertion of the theorem follows. □

Theorem 6.12 and Corollary 6.8 immediately imply the following.

Corollary 6.13. *Let f be strongly subhyperbolic and let $(\widetilde{\mathcal{O}}_f, \mathcal{O}_f)$ be dynamically associated to f. Then*

$$\frac{1}{\tilde{\rho}_f(z)} \leq \mathrm{O}(|z|) \quad as \quad z \to \infty,$$

where $\tilde{\rho}_f(z)$ denotes the density of the hyperbolic metric on $\widetilde{\mathcal{O}}_f$.

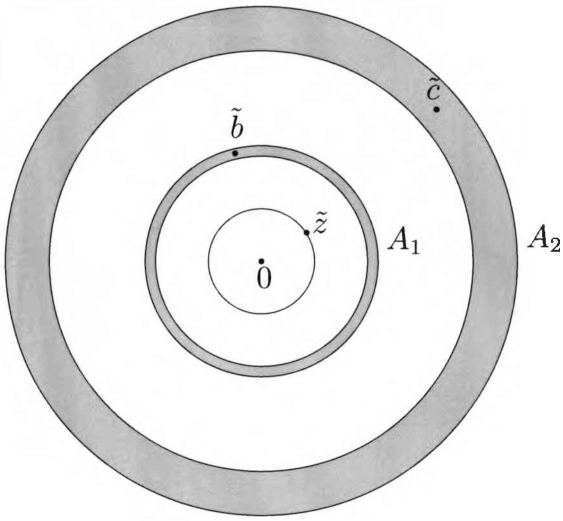

Figure 6.1: The parameters $\tilde{b}$ and $\tilde{c}$ belong to the compact annuli A_1 and A_2.

6.3.3 Proof of uniformity

Using Theorems 6.12 and 6.11, we can finally deduce that $f : \tilde{\mathcal{O}}_f \to \mathcal{O}_f$ is a uniform expansion with respect to the hyperbolic metric of $\mathcal{O}_f$.

Proof of Theorem 6.10. Let $\tilde{\rho}_f$ and ρ_f denote the densities of the hyperbolic metrics on $\tilde{\mathcal{O}}_f$ and $\mathcal{O}_f$, respectively. Since $f : \tilde{\mathcal{O}}_f \to \mathcal{O}_f$ is a covering map, our claim is equivalent to the statement that there is a constant $E > 1$ such that

$$\frac{\tilde{\rho}_f(z)}{\rho_f(z)} \geq E > 1.$$

If $\mathcal{F}(f) \neq \emptyset$, recall that by Proposition 6.7(e), $\mathcal{O}_f$ is modelled on a hyperbolic domain S_f with $\overline{f^{-1}(S_f)} \subset S_f$, implying that $\mathcal{O}_f$ and $\tilde{\mathcal{O}}_f$ have no common boundary points in $\mathbb{C}$. The same is true if the underlying surface is $\mathbb{C}$, which means that ∞ is the only common boundary point of $\mathcal{O}_f$ and $\tilde{\mathcal{O}}_f$. Hence it only remains to check that for some $E' > 1$,

$$\lim_{z\to\infty} \frac{\tilde{\rho}_f(z)}{\rho_f(z)} \geq E' > 1.$$

Let $C \subset \mathcal{O}_f$ be the complement of a closed Euclidean disk centred at 0 such that $\nu_f(z) = 1$ for all $z \in C$, and denote by ρ_C the density of the hyperbolic metric on C. Then there is a right half-plane $H \subset \mathbb{C}$ such that the map $\exp : H \to C,\ z \mapsto \mathrm{e}^z$ is a covering. Hence the asymptotic behaviour of ρ_C is given by

$$\rho_C(z) = \mathrm{O}\left(\frac{1}{|z| \cdot \log|z|}\right) \quad \text{as} \quad z \to \infty.$$

By Theorem 6.5, we have $\rho_C(z) \geq \rho_f(z)$, and so

$$\rho_f(z) \leq \mathrm{O}\left(\frac{1}{|z| \cdot \log|z|}\right) \quad \text{as} \quad z \to \infty.$$

It now follows from Corollary 6.13 that

$$\frac{\tilde{\rho}(z)}{\rho(z)} \geq \mathrm{O}\left(\log|z|\right)$$

and hence

$$\frac{\tilde{\rho}(z)}{\rho(z)} \to \infty \quad \text{as} \quad z \to \infty.$$

□

Remark. If we replace the ramified points by punctures, i.e., if we consider the hyperbolic domain $U : \mathbb{C} \setminus \{z_j\}$ instead of the orbifold $\mathcal{O}$, the same bound for the asymptotic behaviour of the density map near ∞ can be obtained using standard estimates of the hyperbolic metric in the twice-punctured plane [Rem, Lemma 2.1]. As will become clear in the proof of Proposition 6.15, the orbifold for which the set of ramified points is given by $\{2k\pi i : k \in \mathbb{Z}\}$ shows that our estimate is best possible.

6.3.4 Cosine maps

Recall that we say that $F = F_{a,b}$ is a *cosine map*, if it can be written as

$$F_{a,b}(z) = a\,\mathrm{e}^{z} + b\,\mathrm{e}^{-z}$$

for some $a, b \in \mathbb{C}^*$. Note that $F_{a,b} = g \circ f$, where $g(z) = az + b/z$ and $f(z) = \mathrm{e}^z$, hence it is easy to check (with the formula given in Section 2.3 for singular sets of compositions of holomorphic maps) that $F_{a.b}$ has no asymptotic values, and exactly two critical values, namely $v_1 = 2\sqrt{ab}$ and $v_2 = -2\sqrt{ab}$. Furthermore, if $z \in \mathbb{C}$ is a preimage of a critical value, then z is a critical point satisfying $\deg(F_{a,b}, z) = 2$, which implies that v_1 and v_2 are totally ramified. Obviously, every subhyperbolic cosine map is automatically strongly subhyperbolic.

As already mentioned, Schleicher studied landing properties of those cosine maps for which the critical values are strictly preperiodic and proved that for such a map, every point in $\mathbb{C}$ is either on a dynamic ray or the landing point of a dynamic ray [Sch07a]. This result will also follow from Theorem 6.1. Moreover, our proof in the case of strongly subhyperbolic cosine maps is considerably more concise than the proof in the general setting and the proof (of the weaker statement) given in [Sch07a]. The reason is that for strongly subhyperbolic cosine maps, we can compute *explicitly* the required estimates of the metrics of certain dynamically associated orbifolds.

Let us start with a simple observation.

Proposition 6.14. *Let $F = F_{a,b}$ be strongly subhyperbolic but not hyperbolic. Then there exists a point $p \in P_{\mathcal{J}} \setminus S(F)$.*

Proof. Since F is not hyperbolic, it follows that at least one critical value of F, say v_1, belongs to $\mathcal{J}(F)$. Now assume that the claim is wrong, i.e., $P_{\mathcal{J}} = S(F)$. Since $P_{\mathcal{J}}$ is forward invariant, this can only occur if $F(v_1) = v_1$ or v_1 and v_2 form a cycle. However, since v_1 is totally ramified, it would then be a superattracting periodic point of F, contradicting the assumption that $v_1 \in \mathcal{J}(F)$. $\square$

For simplicity, let us assume that $\{v_1, v_2\} \subset \mathcal{J}(F)$; the case when $\mathcal{F}(F) \neq \emptyset$ can be treated in a very similar way (and is even easier). Let $\mathcal{O}_F = (\mathbb{C}, \nu_F)$ and $\widetilde{\mathcal{O}}_F = (\mathbb{C}, \tilde{\nu}_F)$, where

$$\nu_F(w) = \operatorname{lcm}\{\deg(F^n, z), \text{ where } F^n(z) = w\} \quad \text{and} \quad \tilde{\nu}_F(z) = \frac{\nu(F(z))}{\deg(F, z)}.$$

It is straightforward to check that $(\widetilde{\mathcal{O}}_F, \mathcal{O}_F)$ is a pair of orbifolds dynamically associated to F. (In fact, this is how we constructed dynamically associated orbifolds in the proof of Proposition 6.7.) In particular, $\nu_F(z) \in \{1, 2, 4\}$ for all $z \in \mathbb{C}$.

Let us fix a point $p \in P_{\mathcal{J}} \setminus S(F)$. Then p has only regular preimages p_i, for which necessarily $\tilde{\nu}_F(p_i) = \nu_F(p) \in \{2, 4\}$. Since F is $2\pi i$-periodic, the orbifold

$\widetilde{\mathcal{O}}_F$ is holomorphically embedded in the orbifold $\mathcal{O}_0 = (\mathbb{C}, \nu_0)$ defined by

$$\nu_0(z) = \begin{cases} \nu_F(p) & \text{if } w = 2\pi n i \text{ for some } n \in \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, if $\tilde{\rho}_F$ and ρ_0 denote the densities of the hyperbolic metrics on $\widetilde{\mathcal{O}}_F$ and $\mathcal{O}_0$, respectively, then

$$\tilde{\rho}_F(z) > \rho_0(z + \eta)$$

holds for all $z \in \mathbb{C}$, where $\eta \in \mathbb{C}$ is some constant. For ρ_0 we can give the following explicit lower bound.

Proposition 6.15. *The density function $\rho_0(z)$ satisfies*

$$\rho_0(z) \geq \frac{1}{64 + 8 \cdot |\operatorname{Re}(z)|}$$

for all $z \in \mathbb{C}$.

Proof. For all points in the punctured halfplane $\{z \neq 0 : \operatorname{Re}(z) \leq 1/2\} \subset \mathbb{C}\backslash\{0,1\}$, the density ρ_1 of the hyperbolic metric of $\mathbb{C} \setminus \{0,1\}$ can be bounded from below by

$$\frac{1}{\rho_1(z)} \leq C_1 \cdot |z| \cdot (C_2 + |\log|z||),$$

where $C_1 := 2\sqrt{2}$ and $C_2 := 4 + \log(3 + 2\sqrt{2})$ [BP78, p. 476].
Let $\mathcal{O}_2 := (\mathbb{C}^*, \nu_2)$, where $\nu_2(1) = 2$ and $\nu_2(z) = 1$ for all $z \neq 1$. We easily see that the map $p : \mathbb{C}\backslash\{0,1\} \to \mathcal{O}_2,\ z \mapsto -4(z^2 - z)$ is a covering map and hence a local isometry. A simple calculation yields

$$\frac{1}{\rho_2(w)} \leq 2C_1 \cdot |\sqrt{1-w}| \cdot |1 - \sqrt{1-w}| \cdot (C_2 + \log 2 + |\log|1 - \sqrt{1-w}||),$$

where $\rho_2(z)$ is the density of the hyperbolic metric of $\mathcal{O}_2$ and $\sqrt{z}$ denotes the principle branch of the squareroot.
Since the map $\mathcal{O}_0 \to \mathcal{O}_2,\ z \mapsto \mathrm{e}^z$ is a covering map, it follows that

$$\frac{1}{\rho_0(z)} \leq 2C_1 \cdot \frac{|\sqrt{1-\mathrm{e}^z}| \cdot |1 - \sqrt{1-\mathrm{e}^z}|}{|\,\mathrm{e}^z\,|} \cdot (C_2 + \log 2 + |\log|1 - \sqrt{1-\mathrm{e}^z}||)$$

for every $z \in \mathcal{O}_0$. Let us simplify the above expression. We note, by expanding with $|1+\sqrt{1-\mathrm{e}^z}|$, that

$$\frac{|\sqrt{1-\mathrm{e}^z}| \cdot |1-\sqrt{1-\mathrm{e}^z}|}{|\,\mathrm{e}^z\,|} = \frac{|\sqrt{1-\mathrm{e}^z}|}{|1+\sqrt{1-\mathrm{e}^z}|},$$

and one can easily see that the obtained expression is bounded from above by $\sqrt{2}$.

Let us now consider $|\log|1-\sqrt{1-e^z}||$. Since $|\log 1/z| = |\log z|$, it is enough to restrict to the case when $|1-\sqrt{1-e^z}| \geq 1$. Here we get

$$|1-\sqrt{1-\mathrm{e}^z}| \;\leq\; 1+\sqrt{|1-\mathrm{e}^z\,|} \leq \max\{2,\ 2\sqrt{|1-\mathrm{e}^z\,|}\} \leq \max\{2,\ 2|\,\mathrm{e}^z\,|\}$$

and hence

$$|\log|1-\sqrt{1-\mathrm{e}^z}|| \leq \log 2 + |\operatorname{Re}(z)|.$$

Together, these estimates yield the proof. □

Remark. Let $a, b, c \in \mathbb{C}$ and denote by $\mathcal{O}_{a,b,c}$ the $\mathbb{C}$-orbifold with signature $(2,2,2)$, with a, b and c being the ramified points. Then there exists a (unique) Möbius map M mapping 0, 1 and -1 to a, b and c, respectively. Moreover, the map $z \mapsto M(\sin \frac{z}{2} i)$ provides a covering map from $\mathcal{O}_0$ to $\mathcal{O}_{a,b,c}$. This observation enables us to estimate the hyperbolic metric of an arbitrary $\mathbb{C}$-orbifold with signature $(2,2,2)$ using simple calculations, and hence provides an alternative way of proving Theorem 6.12, which — though it is less elegant — uses only elementary observations.

6.4 Construction of a semiconjugacy

Recall that our goal is to construct a continuous and surjective map $\phi : \mathcal{J}(g) \to \mathcal{J}(f)$, where g is any map of disjoint type that belongs to the family

$$\{g_\lambda(z) = f(\lambda z) : \lambda \in \mathbb{C}\},$$

such that

$$f \circ \phi(z) = \phi \circ g(z)$$

holds for all $z \in \mathcal{J}(g)$. Recall that by [Rem, Theorem 5.2], any two such maps g and g' are conjugate on their Julia sets, hence it is enough to prove the statement for one such map. We start with the construction of such a map g. Let us fix a pair of orbifolds $(\widetilde{\mathcal{O}}_f, \mathcal{O}_f)$ dynamically associated to f with underlying surfaces $\widetilde{S}_f$ and S_f, respectively. Note that by Proposition 6.7(c), S_f can be written as $S_f = \mathbb{C} \setminus C$, where C is a, possibly empty, compact set.
Observe that for every $\lambda \in \mathbb{C}^*$, $S(g_\lambda) = S(f)$. Let $K > 0$ be sufficiently large, such that $(P(f) \cup C) \subset \{|z| < K/2\}$. Since f is entire, it maps bounded sets to bounded sets, hence there exists $L \geq K$ such that

$$f^{-1}(\{z : |z| > L\}) \subset \{z : |z| > K + 1\}.$$

Let us fix a constant $L \geq K$ with this property and define $\mu := K/L$. It then follows that if $g = g_\mu$ and z is a point with $|g(z)| > L$, then $|\mu z| > K + 1$ and hence $|z| > L + L/K$. This means,

$$g^{-1}(\{z : |z| > L\}) \subset \{z : |z| > L + L/K\},$$

and, in particular, it follows from Proposition 3.11 that g is of disjoint type.
Define

$$V_j := f^{-j}(\{z : |z| > K\}) \quad \text{and} \quad U_j := g^{-j}(\{z : |z| > L\}).$$

Remark. Note that $V_j \subset \mathcal{O}_f \cap \widetilde{\mathcal{O}}_f$ holds for all $j \geq 0$, such as $U_{j+1} \subset U_j$, since g is of disjoint type. Furthermore, $\mathcal{J}(g)$ is the set of those points that are never mapped into $\mathbb{C} \setminus U_0$, hence $\mathcal{J}(g)$ equals the limit of the domains U_j.

Starting with $\phi_0 \equiv \mathrm{id}$, we want to construct a sequence of conformal isomorphisms

$$\phi_j : U_{j-1} \to V_{j-1}$$

for $j \geq 1$, such that

$$f \circ \phi_{j+1} = \phi_j \circ g.$$

We will define the sequence inductively. Since $\phi_0 \equiv \mathrm{id}$, the map ϕ_1 is given

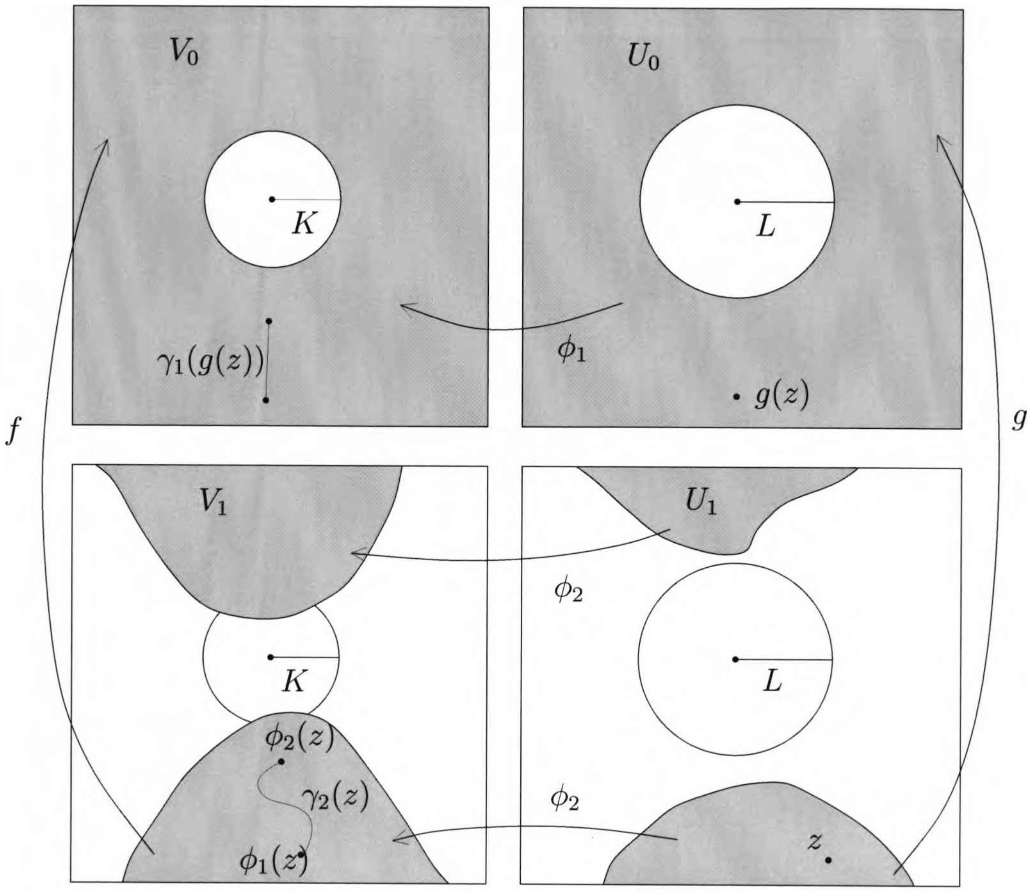

Figure 6.2: The construction of the curve $\gamma_2(z)$ and the isomorphism ϕ_2.

by the formula $\phi_1(z) = \mu z$. For a point $z \in V_0$ let $\gamma_1(z)$ be the straight line segment connecting $z = \phi_0(z)$ and $\mu z = \phi_1(z)$ (we can actually choose γ_1 to be any rectifiable curve which connects z and $\phi_1(z)$ within the domain V_0). To define ϕ_2 at a point $z \in U_1$, we consider the line segment $\gamma_1(g(z)) \subset V_0$. By definition,

$$f^{-1}(\gamma_1(g(z))) \subset V_1.$$

Since $f(\phi_1(z)) = g(z)$, there is a preimage component $\gamma_2(z)$ of $\gamma_1(g(z))$, such that one endpoint of $\gamma_2(z)$ is $\phi_1(z)$. We define $\phi_2(z)$ to be the other endpoint of $\gamma_2(z)$ (see Figure 6.2).

Continuing inductively, we define the curve $\gamma_{j+1}(z) \subset V_j$ to be the pullback of $\gamma_j(g(z)) \subset V_{j-1}$ under f with one endpoint at $\phi_j(z)$, and we define $\phi_{j+1}(z)$ to be the other endpoint $\gamma_{j+1}(z)$.

We want to give some properties of the maps ϕ_j. Since f and g are holomorphic and in particular continuous, each map ϕ_j is continuous as well. By induction, it also follows that each map ϕ_j is injective and surjective. Hence each map ϕ_j is a conformal isomorphism, mapping a component of U_{j-1} onto a component of V_{j-1}.

Theorem 6.16. *The maps $\phi_j|_{\mathcal{J}(g)}$ converge uniformly with respect to the hyperbolic orbifold metric $\rho_f(z)|dz|$ on $\mathcal{O}_f$ to a continuous surjective function*

$$\phi : \mathcal{J}(g) \to \mathcal{J}(f)$$

so that $f \circ \phi = \phi \circ g$. Moreover, $\phi(I(g)) = I(f)$ and $\phi|_{I(g)}$ is a homeomorphism.

Proof. With respect to the hyperbolic metric on $\mathcal{O}_f$, we denote by $d_f(w_1, w_2)$ the distance between two points $w_1, w_2 \in \mathcal{O}_f$, and by $\ell_f(\gamma)$ the length of a rectifiable curve $\gamma \subset \mathcal{O}_f$. Let $z \in U_j$. Since $U_j \subset U_{j-1}$, both ϕ_{j+1} and ϕ_j are defined in a neighbourhood of z and it follows from our construction that

$$d_f(\phi_{j+1}(z), \phi_j(z)) \leq \ell_f(\gamma_{j+1}(z)). \tag{6.8}$$

Since for every point $z \in U_0$,

$$\gamma_1(z) \subset \left(\mathbb{C}\backslash\overline{D_K(0)}\right) \subset \left(\mathbb{C}\backslash\overline{D_{\frac{K}{2}}(0)}\right) \subset \mathcal{O}_f,$$

we obtain an upper bound for $\ell_f(\gamma_1(z))$ by computing its length with respect to the hyperbolic metric in $\mathbb{C}\backslash\overline{D_{\frac{K}{2}}(0)}$, which is given by $(|z|\,(\log|z| - \log(K/2)))^{-1}\,|dz|$. Hence

$$\ell_f(\gamma_1(z)) \leq \log\left(\frac{\log|1/\mu|}{\log|z| - \log(K/2)} + 1\right) \leq \log\left(\frac{\log|1/\mu|}{\log 2} + 1\right) =: \nu$$

Recall that by Lemma 6.6, there is a constant $E > 1$, such that $\|Df(z)\|_{\mathcal{O}_f} \geq E$ holds for all $z \in \widetilde{\mathcal{O}}_f$. Since $\gamma_{j+1}(z) \subset V_j \subset \widetilde{\mathcal{O}}_f$ is obtained as a pullback of $\gamma_1(g^j(z))$ under the map f^j, it follows from equation (6.8) that

$$d_f(\phi_{j+1}(z), \phi_j(z)) \leq \frac{\nu}{E^j}.$$

This means that the maps $\phi_j|_{\mathcal{J}(g)}$ from a Cauchy sequence, and since the orbifold metric is complete, there is a continuous limit function

$$\phi : \mathcal{J}(g) \to \mathcal{O}_f.$$

Note that ϕ necessarily satisfies

$$d_f(\phi(z), z) \leq \sum_{j=0}^{\infty} d_f(\phi_{j+1}(z), \phi_j(z)) \leq \sum_{j=0}^{\infty} \nu \cdot \frac{1}{E^j} = \nu \cdot \frac{E}{E-1} \tag{6.9}$$

as well as

$$f^n(\phi(z)) = \phi(g^n(z)) \tag{6.10}$$

for all $n \in \mathbb{N}$ and all $z \in \mathcal{J}(g)$.

We want to derive some properties of the limit function ϕ. By equation (6.9),

$$\phi(z_n) \to \infty \quad \text{if and only if} \quad z_n \to \infty, \tag{6.11}$$

so together with equation (6.10) this implies that $\phi(I(g)) \subset I(f)$. In particular, it follows that $\phi(\mathcal{J}(g)) \subset \mathcal{J}(f)$, since $\mathcal{J}(g) = \overline{I(g)}$ and $\mathcal{J}(f) = \overline{I(f)}$ (Corollary 2.9). Let $w \in I(f)$. Then $w \in V_j$ for all sufficiently large j, so we can consider the sequence $z_j := \phi_j^{-1}(w)$. If z is an accumulation point of (z_j), then $\phi(z) = w$ (note that by relation (6.11), $z \neq \infty$). Hence, $\phi : I(g) \to I(f)$ is surjective.

Next we will show that $\phi|_{I(g)}$ is injective. So let $\mathscr{S}$ be a static partition of f and let $z, \tilde{z} \in I(g)$ be two points such that $\phi(z) = \phi(\tilde{z}) =: w$. By definition, $\phi_j(z), \phi_j(\tilde{z}) \to w$, and it follows from the inductive definition of the maps ϕ_j that for every sufficiently large j, there exists a fundamental domain $F_j \in \mathscr{S}$ such that $g^j(z), g^j(\tilde{z}) \in F_j$. On the other hand, it follows from equation (6.10) that $\phi(g^j(z)) = \phi(g^j(\tilde{z}))$ holds for all $j \in \mathbb{N}$. Furthermore, equation (6.9) implies that

$$d_f(g^j(z), g^j(\tilde{z})) \leq d_f(g^j(z), \phi(g^j(z))) + d_f(\phi(g^j(\tilde{z})), g^j(\tilde{z})) \leq 2\nu \cdot \frac{E}{E-1}.$$

By standard expansion estimates (see e.g. [Rem, Lemma 2.7]), the distance between $g^j(z)$ and $g^j(\tilde{z})$ must be unbounded, unless the points z and $\tilde{z}$ are equal, implying that ϕ is injective.

Observe that by equation (6.11), ϕ can be extended (sequentially) continuously to a map $\widehat{\phi}: \widehat{\mathcal{J}(g)} \to \widehat{\mathcal{J}(f)}$ with $\widehat{\phi}(\infty) = \infty$. Hence $\widehat{\phi}\left(\widehat{\mathcal{J}(g)}\right)$ is the continuous image of a compact set, hence it is compact and $\widehat{\phi}(\mathcal{J}(g)) = \phi(\mathcal{J}(g))$ is closed. So

$$I(f) = \phi(I(g)) \subset \phi(\mathcal{J}(g)) \subset \mathcal{J}(f) = \overline{I(f)}$$

and since $\phi(\mathcal{J}(g))$ is closed, it follows that $\phi(\mathcal{J}(g)) = \mathcal{J}(f)$, hence ϕ is surjective. □

Since the restriction of the map ϕ in Theorem 6.16 to the escaping set of the disjoint type map is a homeomorphism, we obtain the following result as an immediate consequence of Theorem 6.16 and Theorem 3.12.

Corollary 6.17. *The escaping set of a strongly subhyperbolic map is disconnected.*

Remark. Dierk Schleicher kindly pointed out that the escaping set of the strongly subhyperbolic map $z \mapsto \pi \sinh z$ is obviously disconnected: the imaginary axis consists of points with bounded orbits and it disconnects the escaping set. (Details on this special function will be given in Section 6.5.)

In terms of dynamic rays, our main result implies the following topological description of the Julia set of certain strongly subhyperbolic maps.

Corollary 6.18. *Let $f \in R^3S$ be a strongly subhyperbolic map. Then $\mathcal{J}(f)$ is a* pinched *Cantor bouquet, consisting of dynamic rays of f and their endpoints. In particular, all dynamic rays of f land and every point in $\mathcal{J}(f)$ is either on a dynamic ray or the landing point of a dynamic ray of f.*

By a pinched Cantor bouquet we mean a quotient of a Cantor bouquet by a closed equivalence relation on its endpoints.

Proof. Let g and $\phi : \mathcal{J}(g) \to \mathcal{J}(f)$ be maps as in Theorem 6.16. By Theorem 3.13, $\mathcal{J}(g)$ is a Cantor bouquet, and this Cantor bouquet consists of dynamic rays of g and their endpoints (see also [RRRS, Theorem 4.7]). Since ϕ is surjective and since its restriction to the Cantor bouquet $\mathcal{J}(g)$ without its endpoints is a homeomorphism, $\mathcal{J}(f)$ must be a pinched Cantor bouquet. □

6.5 Model of the dynamics of a map f with $\mathcal{J}(f) = \mathbb{C}$

This section is dedicated to the description of the topological dynamics of the function

$$f(z) := \pi \sinh z.$$

We will define a "simple" model consisting of a topological space $\overline{X}$ and a map $\mathcal{M} : \overline{X} \to \overline{X}$ such that if g is any map of disjoint type in the family $g_\lambda : z \mapsto \lambda \sinh z$ then

- $\mathcal{J}(g)$ is homeomorphic to $\overline{X}$, and
- $\mathcal{M}|_{\overline{X}}$ is conjugate to $g|_{\mathcal{J}(g)}$.

We will transfer the ideas from [Rem06] where such a model was constructed for exponential maps whose singular value belongs to some attracting basin. The adoption of [Rem06] to the maps we are interested in is particularly simple since in left and right half-planes, sufficiently far away from the imaginary axis, any map g_λ with $\lambda > 0$ is essentially the same (i.e., up to a constant factor) as $z \mapsto e^{-z}$ and $z \mapsto e^z$, respectively. For this reason, we will skip the details and refer to [Rem06] as well as the extensive work on dynamics of cosine maps by Rottenfußer and Schleicher [RS08] for further consideration.
Once we have constructed such a model for a disjoint type map $g \in \{g_\lambda\}$, Theorem 6.16 tells us that there is a semiconjugacy between g and f on their Julia sets, and hence also between the model map $\mathcal{M}$ and f. The combinatorial dynamics of f on $\mathcal{J}(f)$ was already established in [Sch07a, Sch07b] and we will summarize here the required results.

6.5.1 Dynamics within the one-parameter family

So let us consider the family

$$g_\lambda(z) := \lambda \sinh z$$

with $\lambda > 0$ (hence $f = g_\pi$). The critical values of g_λ are $\pm\lambda i$. Every map $g_\lambda : \mathbb{R} \to \mathbb{R}$ is a homeomorphism with $g_\lambda(0) = 0$ and $\mathbb{R} \setminus \{0\} \subset I(g_\lambda)$. Furthermore, $g_\lambda(i\mathbb{R}) \subset [-\lambda i, \lambda i]$.

Both critical values of f are mapped by f to the repelling fixed point 0. Since the maps g_λ have no asymptotic values, the postsingular set of f equals $\{\pm\pi i, 0\}$. Hence f is postsingularly finite and strongly subhyperbolic. Furthermore, $\mathcal{J}(f) = \mathbb{C}$.

For $\lambda > 0$ chosen sufficiently small, the origin is an attracting fixed point and the subinterval $[-\lambda i, \lambda i]$ of the imaginary axis is mapped into itself and thus belongs to the immediate basin of attraction of 0. Hence by choosing λ sufficiently small, we obtain a map g_λ of disjoint type (see Proposition 3.11). From now on, we will fix $\lambda_0 > 0$ such that the corresponding map $g_{\lambda_0} =: g$ is of disjoint type.

Note that for every $n \in \mathbb{Z}$, the horizontal line

$$L_n := \{z : \operatorname{Im} z = (n + 1/2)\pi\}$$

is mapped by g (or any other g_λ with $\lambda \in \mathbb{R}$) to $i\mathbb{R}\backslash[-\lambda_0 i, \lambda_0 i]$, hence $\mathcal{J}(g)$ is contained in horizontal half-strips which are the components of $\mathbb{C}\backslash\left(\bigcup_{n\in\mathbb{Z}} L_n \cup i\mathbb{R}\right)$. This means that each point $z \in \mathcal{J}(g)$ is contained in a domain

$$\begin{aligned} S_{n_L} &:= \{z : \operatorname{Re} z < 0, \operatorname{Im} z \in ((n - 1/2)\pi, (n + 1/2)\pi)\} \text{ or} \\ S_{n_R} &:= \{z : \operatorname{Re} z > 0, \operatorname{Im} z \in ((n - 1/2)\pi, (n + 1/2)\pi)\}. \end{aligned}$$

Note that the domains S_{n_L} and S_{n_R} are very similar to fundamental domains of g (see Definition 2.12). In fact, this partition allows us the same combinatorial approach as the standard static partitions. For instance, the restriction of g (or any other g_λ with $\lambda \in \mathbb{R}$) to any of the half-strips is a conformal isomorphism onto its image which is the left or right half-plane. The reason for choosing this particular partition of the plane is the possibility to give a very simple description of the "pinching", as we will see later.

6.5.2 Topological model

Let $\mathscr{S}^{\mathbb{N}} := (\mathbb{Z}_L \cup \mathbb{Z}_R)^{\mathbb{N}}$ be the space of infinite sequences of elements in $\mathbb{Z}_L \cup \mathbb{Z}_R$, where $\mathbb{Z}_L := \{\ldots, -1_L, 0_L, 1_L, \ldots\}$ and $\mathbb{Z}_R := \{\ldots, -1_R, 0_R, 1_R, \ldots\}$ are two disjoint copies of $\mathbb{Z}$. By the previous argument, we can assign to a point $z \in \mathcal{J}(g)$ a unique sequence $\underline{s} = s_0 s_1 \ldots \in \mathscr{S}^{\mathbb{N}}$ defined by $g^n(z) \in S_{s_n}$. We will call such a sequence the *external address* of z, due to the similarity between the halfstrips S_{s_n} and fundamental domains of g.

For every $i \in \mathbb{Z}$ we define $|i_L| := |i| =: |i_R|$. Furthermore, since $\mathcal{J}(g)$ consists of (asymptotically horizontal) dynamic rays and their endpoints [RS08, Theorem 4.1], our model $\overline{X}$ should be a subset of the space

$$\mathscr{S}^{\mathbb{N}} \times [0, \infty).$$

Note that the relation $\dots i_L < i_R < (i+1)_L < \dots$ defines an order on $\mathscr{S}^{\mathbb{N}}$. Thus $\mathscr{S}^{\mathbb{N}} \times [0,\infty)$ is equipped with the product topology of the topology on $\mathscr{S}^{\mathbb{N}}$ (induced by the order relation) and the standard topology on $\mathbb{R}$.
Let $(\underline{s}, t)$ be a point in $\mathscr{S}^{\mathbb{N}} \times [0,\infty)$. We should think of the first entry s_0 in $\underline{s}$ as the imaginary part of the point (or its height corresponding to our horizontal strips), together with the information whether it is lying left or right from the imaginary axis. The second entry t should be thought of as the absolute value of the real part of the point. Hence it is helpful to think of a point $(\underline{s}, t) \in \mathscr{S}^{\mathbb{N}} \times [0,\infty)$ in its "complexified" version $C(\underline{s}, t) := t + \pi i s_0$. Let us denote by $T(\underline{s}, t) := t$ the projection onto the second coordinate. We can now define our model map to be

$$\mathcal{M} : \mathscr{S}^{\mathbb{N}} \times [0,\infty) \to \mathscr{S}^{\mathbb{N}} \times [0,\infty),\ (\underline{s}, t) \mapsto (\sigma(\underline{s}), F(t) - \pi|s_1|),$$

where σ denotes the one-sided shift map and $F(t) := \mathrm{e}^t - 1$ denotes the standard model map for exponential growth.
Recall that the maps we consider behave like the exponential in each of the halfplanes. The essential characteristic of our model map now is that as for exponential maps, the size of the image $|C(\mathcal{M}(\underline{s}, t))|$ of a point $(\underline{s}, t)$ is roughly the exponential of its real part. More precisely, $F(t)/\sqrt{2} \leq |C(\mathcal{M}(\underline{s}, t))| \leq F(t)$ whenever $T(\underline{s}, t) \geq 0$. Hence we define the model sets $\overline{X}$ and X to be

$$\begin{aligned} \overline{X} &:= \{(\underline{s}, t) \in \mathscr{S}^{\mathbb{N}} \times [0,\infty) : T(\mathcal{M}^n(\underline{s}, t)) \geq 0 \text{ for all } n \geq 0\} \text{ and} \\ X &:= \{(\underline{s}, t) \in \overline{X} : T(\mathcal{M}^n(\underline{s}, t)) \to \infty \text{ as } n \to \infty\}. \end{aligned}$$

By [Rem06, Observation 3.1], $\overline{X}$ is homeomorphic to a straight brush. In particular, for every external address $\underline{s}$ there exists a unique $t_{\underline{s}} \in [0, \infty]$ such that $\{t \geq 0 : (\underline{s}, t) \in \overline{X}\} = [t_{\underline{s}}, \infty)$. We denote by $E(\overline{X}) := \{(\underline{s}, t_{\underline{s}})\}$ the set of *endpoints* of $\overline{X}$.
By iterating forward under the model map $\mathcal{M}$ and backwards under g, we obtain

a sequence of maps that converges to a homeomorphism $\Phi : \overline{X} \to \mathcal{J}(g)$ such that

$$\Phi \circ \mathcal{M}(z) = g \circ \Phi(z)$$

for all $z \in \overline{X}$. The key argument for such a limit to exist is again uniform hyperbolic contraction of the map g and the fact that the mapping behaviour of the model map reflects that of g. (For a precise statement see [Rem06, Section 3] or [RS08, Proposition 3.3].) A proof of the above statement is essentially the same as in the case of exponential maps in [Rem06, Theorem 9.1], which is why we skip the details here. A proof can also be derived by essentially the same estimates as given in the proof of Theorem 6.16.

By Theorem 6.16 there is a surjective map $\phi : \mathcal{J}(g) \to \mathcal{J}(f)$ such that $f(\phi(z)) = \phi(g(z))$ holds for all $z \in \mathcal{J}(g)$. Moreover, ϕ restricts to a homeomorphism between $I(g)$ and $I(f)$. As already mentioned, every point $z \in I(f)$ escapes within the strips S_{s_i} with $s_i \in \mathbb{Z}_L \cup \mathbb{Z}_R$, since the forward orbit of any point in the boundary of the strips has a bounded orbit. Recall from the proof of Theorem 6.1 that, by choosing the inverse branches of the maps f^n appropriately, the conjugacy ϕ relates the escaping points of g and f with respect to the combinatorics in terms of their external addresses. From Corollary 6.18, we obtain that $\mathcal{M}$ projects to a function $\widetilde{\mathcal{M}}$ on $\widetilde{X} := \overline{X}/\sim_p$, where $\sim_p$ is an equivalence relation on the set $E(\overline{X})$ of endpoints of $\overline{X}$, such that $\widetilde{\mathcal{M}} : \widetilde{X} \to \widetilde{X}$ is conjugate to $f : \mathcal{J}(f) \to \mathcal{J}(f)$. The equivalence relation $\sim_p$ tells us which dynamic rays are being "pinched". We will now describe $\sim_p$ explicitly using the results from [Sch07a, Sch07b].

6.5.3 Combinatorial description

For every $n \in \mathbb{Z}$ we set

$$\begin{aligned} U_{(n,0)} &:= \{z : \operatorname{Re} z < 0, \operatorname{Im} z \in ((2n\pi, 2(n+1)\pi)\} \text{ and} \\ U_{(n,1)} &:= \{z : \operatorname{Re} z > 0, \operatorname{Im} z \in (2n\pi, 2(n+1)\pi)\}. \end{aligned}$$

One can easily see that the restrictions $f : U_{(n,0)} \to \mathbb{C} \setminus (\mathbb{R}^+ \cup [-\pi i, \pi i])$ and $f : U_{(n,1)} \to \mathbb{C} \setminus (\mathbb{R}^- \cup [-\pi i, \pi i])$ are conformal isomorphisms. While the halfstrips S_{s_n} play the role of fundamental domains, the sets $U_{(n,k)}$ are very similar to itinerary domains, since the relevant part of the boundary of every $U_{(n,k)}$ is a prefixed dynamic ray. In order to maintain the analogy, we will call a sequence $(n_0, k_0)(n_1, k_1) \ldots \in (\mathbb{Z} \times \{0,1\})^{\mathbb{N}}$ an *itinerary*. If γ is a dynamic ray such that for every $i \geq 0$ there exists a domain $U_{(n_i,k_i)}$ with $f^i(\gamma) \subset U_{(n_i,k_i)}$ then we assign to γ the (well-defined) itinerary $\operatorname{itin}(\gamma) = (n_0, k_0)(n_1, k_1) \ldots$. Since a dynamic ray of f is either contained in some half-strip $U_{(n,k)}$ or is completely contained in the boundary of such a domain, it follows that an itinerary cannot be assigned to a ray γ if and only if there is $n \geq 0$ such that $f^n(\gamma)$ equals $\mathbb{R}^+$ or $\mathbb{R}^-$, or equivalently, if $s_{n+j} \equiv 0_R$ or 0_L for all $j \geq 0$, where $\underline{s} = s_0 s_1 \ldots$ is the external address of γ. This means that to every external address $\underline{s}$ in

$$\mathcal{S}_+^{\mathbb{N}} := \{\underline{s} : t_{\underline{s}} < \infty\} \setminus \{\underline{s} : s_{n+j} \equiv 0_R \text{ or } 0_L \text{ for some } n \geq 0 \text{ and all } j \geq 0\}$$

we can assign a unique itinerary $\operatorname{itin}(\underline{s}) := \operatorname{itin}(\gamma_{\underline{s}})$. Let us first comment on those external addresses that belong to

$$\mathcal{S}_-^{\mathbb{N}} := \{\underline{s} : t_{\underline{s}} < \infty\} \setminus \mathcal{S}_+^{\mathbb{N}}.$$

The mapping behaviour of the map f is fairly simple and allows us to describe completely all tuples and quadruples of external addresses in $\mathcal{S}_-^{\mathbb{N}}$ for which the respective dynamic rays land together. For instance, for all addresses $\underline{s}^i$ that belong to either the left of right quadruple

$$s_0 \ldots s_j \begin{cases} (2m)_R \; (2n+1)_R \; 1_L \; \overline{0_R} \\ (2m)_R \; (2n)_R \; 1_R \; \overline{0_L} \\ (2m+1)_R (2n+1)_L \; 1_R \; \overline{0_L} \\ (2m+1)_R \; (2n))_L \; 1_L \; \overline{0_R} \end{cases} \qquad s_0 \ldots s_j \begin{cases} (2m+1)_L (2n+1)_R \; 1_L \; \overline{0_R} \\ (2m+1)_L \; (2n)_R \; 1_R \; \overline{0_L} \\ (2m)_L \; (2n+1)_L \; 1_R \; \overline{0_L} \\ (2m)_L \; (2n))_L \; 1_L \; \overline{0_R} \end{cases}$$

where $m \in \mathbb{Z}$ and $n \geq 0$ are fixed, we define $(\underline{s}^i, t_{\underline{s}^i}) \sim_p (\underline{s}^j, t_{\underline{s}^j})$. We will not list the remaining combinations since there are not so many of them and each one is easy to determine using elementary computations.

The remaining task is to determine those external addresses in $\mathcal{S}_+^{\mathbb{N}}$ such that the corresponding dynamic rays land together. Let γ be a dynamic ray with external address $\underline{s} \in \mathcal{S}_+^{\mathbb{N}}$ and let w be its landing point.

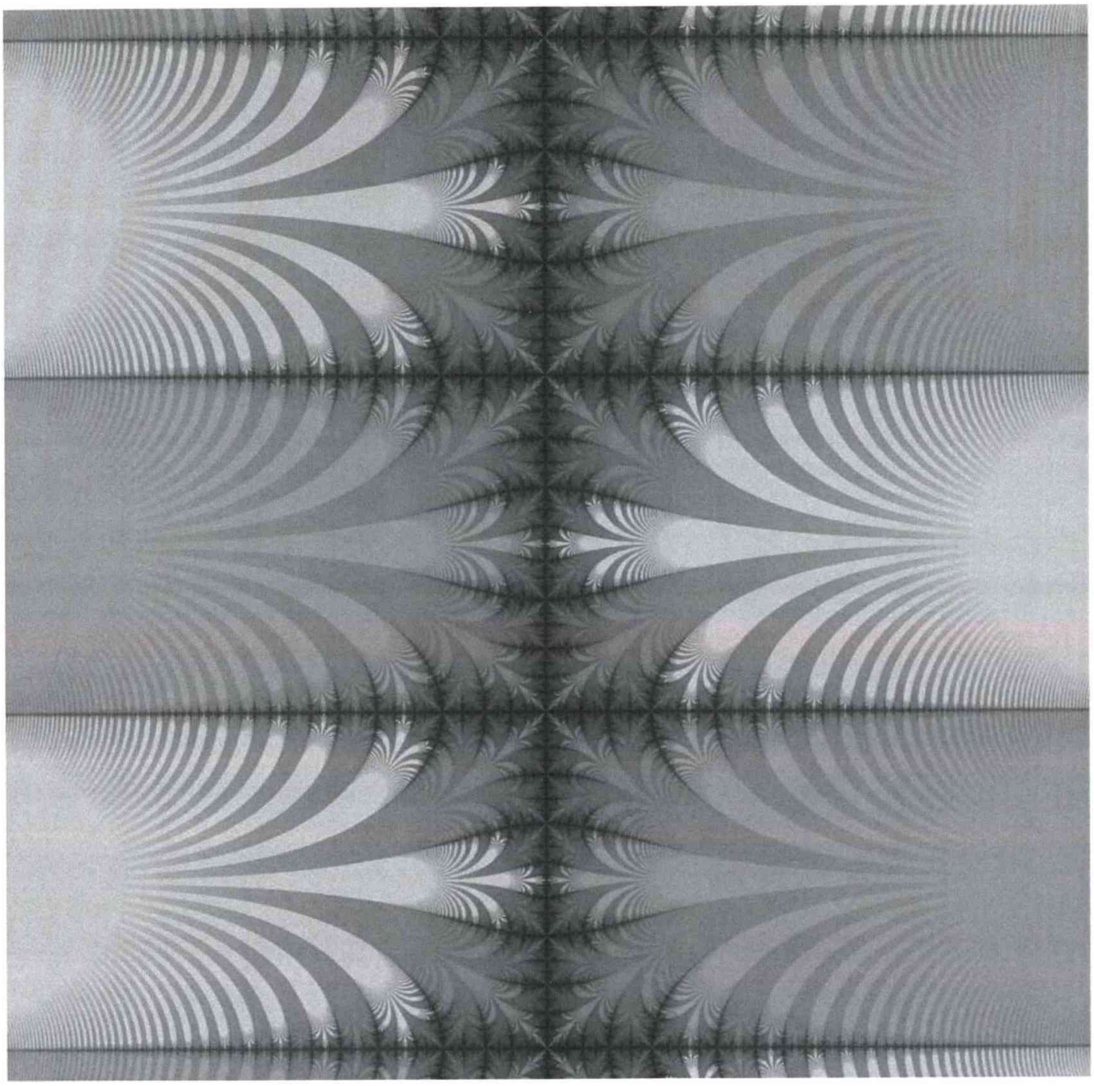

Figure 6.3: The Julia set of the map $f(z) = \pi \sinh z$, showing its structure of a pinched Cantor bouquet. This wonderful picture was kindly provided by Arnaud Chéritat, who produced it (after my talk on this topic) at a conference in Toulouse in June 2009.

Then either g is the only dynamic ray that lands at w or there is exactly one more such dynamic ray [Sch07b]; the latter case occurs if and only if w is eventually mapped into $[\pi i, -\pi i]$ (and remains there without ever being mapped to 0). Let $\underline{s}, \tilde{\underline{s}}$ be the external addresses of two dynamic rays landing at the same point w. It follows that $\operatorname{itin}(\underline{s})$ and $\operatorname{itin}(\tilde{\underline{s}})$ must be of the form

$$(n_0,k_0)\dots(n_j,k_j)\begin{cases}(n_{j+1},\ \ k_{j+1}\ \ )(0,\ \ k_{j+2}\ )(0,\ \ k_{j+3}\ \ )\dots\\(n_{j+1},1-k_{j+1})(0,1-k_{j+2})(0,1-k_{j+3})\dots\end{cases} \tag{6.12}$$

or with -1 instead of 0.

On the other hand, it follows from [Sch07a, Lemma 5] and elementary computations that two dynamic rays with itineraries as in equation (6.12) do indeed land together: such dynamic rays have a forward image that lands in the interval $[-\pi i, \pi i]$ and its landing point is never mapped to 0. So let $\underline{s}, \tilde{\underline{s}} \in \mathscr{S}_+^{\mathbb{N}}$. It follows that $(\underline{s}, t_{\underline{s}}) \sim_p (\tilde{\underline{s}}, t_{\tilde{\underline{s}}})$ if and only if $\operatorname{itin}(\underline{s})$ and $\operatorname{itin}(\tilde{\underline{s}})$ are of the form given by equation (6.12) (or with -1 instead of 0).

Remark. One can certainly relate the model $(\overline{X}, \mathcal{M})$ directly to $\mathcal{J}(f)$. The reason to incorporate a disjoint type map is simply to show what to do when the considered strongly subhyperbolic map f can be embedded in a family where the topological dynamics of disjoint type maps is well understood.

6.6 Questions and remarks

Let us recall an important trick that we used in this chapter. For the construction of the orbifold $\mathcal{O}_f$, we picked a repelling fixed point $p \notin S(f)$ and assigned to p the ramification value $2k$, $k \geq 1$. The effect was that $\widetilde{\mathcal{O}}_f$ contained a sequence (z_i) of points with the same ramification value $2k$, such that

$$|z_i| < |z_{i+1}| \leq K|z_i| \tag{6.13}$$

for some $K > 1$. Now, as we have seen in the proof of Proposition 4.5, the estimate in equation (6.13) occurs regularly for subsequences of the preimages of an arbitrary point $w \notin S(f)$. Hence without the initial ramification of a point $p \in \mathcal{O}_f$, meaning by ramifying only the points in $P_{\mathcal{J}}$, the corresponding orbifold $\widetilde{\mathcal{O}}_f$ would still contain a sequence of ramified points that behave as in equation (6.13), but their ramification values would not necessarily be equal. Hence this additional step would be needless if the following question — as we believe is plausible — could be answered affirmatively.

Question 6.19. *Let $\mathcal{O} = (S, \nu)$ and $\widetilde{\mathcal{O}} = (S, \tilde{\nu})$ be hyperbolic orbifolds for which* *there exists a unique point w such that*

$$\nu(z) = \begin{cases} \tilde{\nu}(z) & \text{if } z \neq w, \\ \tilde{\nu}(z) + 1 > 2 & \text{if } z = w. \end{cases}$$

Let ρ and $\tilde{\rho}$ denote the metric densities on $\mathcal{O}$ and $\widetilde{\mathcal{O}}$, respectively. Does

$$\rho(z) \geq \tilde{\rho}(z)$$

hold for all $z \in S$?

Now let us discuss our restrictions to strongly subhyperbolic maps. A subhyperbolic map f can fail to be strongly subhyperbolic because of one of the following two reasons: $\mathcal{J}(f)$ contains an asymptotic value of f or there is a sequence of critical points in $\mathcal{J}(f)$ with unbounded local degree.
Consider the maps $E_1(z) := \frac{1}{e^2}e^z$ and $E_2(z) := 2\pi i e^z$. The function E_1 is of disjoint type while E_2 is subhyperbolic, since the asymptotic value 0 is prefixed under E_2. It is known that E_1 and E_2 are not topologically conjugate on their escaping sets [Rem06, Proposition 2.1]. In fact, for E_1, all dynamic rays have a landing point in $\mathbb{C}$, while for E_2 there are uncountably many dynamic rays, each of which accumulates everywhere upon itself [Rem07b]. Hence Theorem 6.1 is not true under the general allowance of asymptotic values in the Julia set. We believe that, with possible minor restrictions, the following should be true.

Conjecture 6.20. *Let f be a subhyperbolic map for which $\mathcal{J}(f) \cap A(f) \neq \emptyset$. Then there is no disjoint type map $g : z \mapsto f(\lambda z)$ such that f and g are conjugate on their escaping sets.*

By the results of Rempe [Rem], the map g in Conjecture 6.20 can be replaced by any disjoint type map in the same parameter space as g. (For a precise notion of the (natural) parameter space of a transcendental entire map in class $\mathcal{B}$, see e.g. [EL92, Section 3], [Rem, Section 2] or Chapter 7.4.1.)
We also believe that dynamic rays which do not land occur in a far more general setting, beyond explicit families of maps.

Conjecture 6.21. *Let $f \in R^3S$ be a geometrically finite map with $\mathcal{J}(f) \cap A(f) \neq \emptyset$. There exists a dynamic ray γ of f whose accumulation set on $\widehat{\mathbb{C}}$ is an indecomposable continuum. In particular, g does not land.*

The methods in [Rem07b] suggest that Conjecture 6.21 should hold in an even more general setting: what seems crucial is the accessibility of an asymptotic value in $\mathcal{J}(f)$.
However, we have no indication of what to expect for maps whose Julia sets contain no asymptotic values but sequences of points with unbounded local degree. The reason for our restriction in Theorem 6.1 is the fact that our methods do not apply in this case (see Proposition 6.9).

Question 6.22. *Let f be subhyperbolic, let $\mathcal{J}(f) \cap A(f) = \emptyset$ and assume that* $\sup_{z \in \mathcal{J}(f)} \deg(f, z) = \infty$. *Do all dynamic rays of f land?*

It would be very interesting to explore this problem, in particular since there are prominent examples of such maps. We want to be more explicit and give an example of a map Φ that is subhyperbolic, has no asymptotic values but such that the local degree at points in $\mathcal{J}(\Phi)$ is unbounded.

6.6.1 Subhyperbolic Poincaré maps

Let p be a complex polynomial of degree $d \geq 2$ and let z_0 be a repelling fixed point of p with multiplier μ. By Poincaré's Theorem [Poi90, Val54], there exists an entire map Φ, which is called a *Poincaré function of p at z_0*, such that the functional equation

$$\Phi(\mu \cdot z) = p(\Phi(z)) \tag{6.14}$$

is satisfied for all $z \in \mathbb{C}$. Now let

$$p(z) = z^2 - 1$$

and let Φ_0 denote a Poincaré function of p at the point $z_0 = (1 + \sqrt{5})/2$. The unique finite critical point of p is 0. Since $p(0) = -1$ and $p(-1) = 0$, the cycle $\{0, -1\}$ is superattracting and, in particular, $P(p) \cap \mathbb{C} = \{0, -1\}$. Note that by Theorem 2.11, p has no other attracting or parabolic cycles in $\mathbb{C}$. Observe also that p has no exceptional values (points with a finite backward orbit) in $\mathbb{C}$; it is now not hard to check that $C(\Phi_0) = P(p) = \{0, -1\}$ (see e.g. [DO08, Theorem 2.10]).
The multiplier of the repelling fixed point z_0 is given by $\lambda = p'(z_0) = 1 + \sqrt{5}$.

By [Val54, p. 160], the order of Φ_0 is given by the formula

$$\rho(\Phi_0) = \frac{\log 2}{\log|\lambda|} < \frac{\log 2}{\log 3} < 1$$

and it follows then from the Denjoy-Carleman-Ahlfors Theorem [Nev53, XI, §4, p.313] that Φ_0 has at most one finite asymptotic value. Let us assume that $A(\Phi_0) \neq \emptyset$ and let w be the unique asymptotic value of Φ_0. By [DO08, Theorem 1], w is an attracting periodic point of p, hence either $w = 0$ or $w = -1$. Let $\gamma(t)$ be an asymptotic path for w, i.e., $\lim_{t\to\infty}\gamma(t) = \infty$ and $\lim_{t\to\infty}\Phi_0(\gamma(t)) = w$. Then

$$\lim_{t\to\infty}\Phi_0(\lambda\gamma(t)) = p(\lim_{t\to\infty}\Phi_0(\gamma(t))) = p(w) \neq w,$$

hence $\gamma_\lambda(t) := \lambda\cdot\gamma(t)$ is an asymptotic path of Φ_0 leading to the asymptotic value $p(w)$. But this contradicts the fact that $A(\Phi_0) = \{w\}$, and hence $A(\Phi_0) = \emptyset$. Since 0 is a critical value of Φ_0, there exists a point $\tilde{z}$ such that

$$\Phi_0(\tilde{z}) = 0 \quad \text{and} \quad \Phi_0'(\tilde{z}) = 0.$$

Let $z_n := \lambda^n\tilde{z}$. Using the functional equation (6.14), it is not difficult to see that for every $n \in \mathbb{N}$,

$$\frac{d^k}{dz^k}\Phi_0(z)|_{z=z_n} = 0 \quad \text{for all } 0 \leq k \leq n,$$

hence for every $n \in \mathbb{N}$, we have $\deg(\Phi_0, z_n) \geq n$.
Let $z' \in \mathbb{C}$ be a point with $\Phi_0(z') = 1$. (Note that such a point exists since Φ_0 has no omitted values.) Then $\Phi_0(\lambda z') = p(\Phi_0(z')) = 0$ and $\lambda\Phi_0'(\lambda z') = 2\Phi_0(z')\Phi_0'(z') = 2\Phi_0(z')$. Since z' is not a critical point of Φ_0, it follows that $a := \lambda z'$ is a regular preimage of 0 under Φ_0.
Let b be a preimage of -1, chosen sufficiently large such that $|a-b|\cdot|\Phi_0'(a)| > 1$. Now consider the map

$$\Phi(z) := (a-b)\cdot\Phi_0(z) + a.$$

Note that $A(\Phi) = \emptyset$, since Φ and Φ_0 differ only by postcomposition with a conformal map. It follows also immediately that $C(\Phi) = \{a, b\}$, $\Phi(a) = a$

and $\Phi(b) = b$, hence Φ is postsingularly finite and in particular subhyperbolic. Moreover, since $|\Phi'(a)| = |a - b| \cdot |\Phi_0'(a)| > 1$, the critical value a is a repelling fixed point of Φ and hence belongs to $\mathcal{J}(\Phi)$, implying that Φ is not hyperbolic. Finally note that the points z_n are mapped to a under Φ satisfying $\deg(\Phi, z_n) \geq n$, so Φ is not strongly subhyperbolic.
Altogether, this means that $\Phi(z)$ is subhyperbolic, $A(\Phi) = \emptyset$ but for every $n \in \mathbb{N}$ there exists a point $z_n \in \mathcal{J}(\Phi)$ such that $\deg(\Phi, z_n) \geq n$, yielding the desired example.

Remark. It is not hard to see that one can use the same idea to construct many more Poincaré maps (corresponding to hyperbolic polynomials of arbitrary large degree) with the desired properties.

7 Nonescaping-hyperbolic components

In this chapter, we consider the space $\mathrm{Hol}^*(\mathbb{C})$ of all entire functions that are not constant or linear. In Section 3.4 we introduced nonescaping-hyperbolic functions, i.e., maps f for which $S(f)$ is bounded and every point in $S(f)$ converges to an attracting cycle of f (in $\mathbb{C}$). The broader theme of this part of the thesis is to understand how nonescaping-hyperbolic maps behave under small perturbations. It is known that within the space of all polynomials or the space of all transcendental entire maps with finitely many singular values, nonescaping-hyperbolic functions exhibit particularly simple and stable dynamics [MSS83, EL92].

The space $\mathrm{Hol}^*(\mathbb{C})$ is naturally equipped with the topology of locally uniform convergence. However, this topology is not convenient for dynamical considerations since maps that are nearby in the corresponding metric often have completely different dynamics (see Example 7.6). We will introduce a new metric χ_{dyn} on $\mathrm{Hol}^*(\mathbb{C})$ which is dynamically more sensible in the sense that it combines locally uniform distance *and* the Hausdorff distance between the sets of singular values. We will say that (families of) maps converge *dynamically* if, roughly speaking, they converge with respect to the metric χ_{dyn} (the precise definition will be given later).

It is certainly difficult to make useful statements about the set of *all* nonescaping-hyperbolic maps, simply because $\mathrm{Hol}^*(\mathbb{C})$ is enormously large. Hence we will focus on certain "slices" of this space. More precisely, we want to study the following question: Let $(\mathscr{F}_n)$ be one-parameter families of entire maps converging dynamically to a family $\mathscr{F}_\infty$. How do the nonescaping-hyperbolic parameters in the respective parameter spaces relate to each other? We show that, under the assumption of a "holomorphic parametrization", the nonescaping-hyperbolic components are converging as kernels. More precisely, let M be a complex manifold. For every $n \in \mathbb{N} \cup \{\infty\}$, let $\mathscr{F}_n = \{f_{n,\lambda}\} \subset \mathrm{Hol}^*(\mathbb{C})$ be a family of entire functions that depend holomorphically on $\lambda \in M$. Furthermore, assume that for every n, the singular values of all maps in $\mathscr{F}_n$ form bounded sets and are holomorphically parametrized by M, and that $\mathscr{F}_n \to \mathscr{F}_\infty$ dynamically.

Theorem 7.1. *If $\widetilde{H}$ is a kernel of a sequence of nonescaping-hyperbolic components of $\mathscr{F}_n$, then exactly one of the following statements holds:*

(*i*) *The map $f_\lambda \in \mathscr{F}_\infty$ is not nonescaping-hyperbolic for any $\lambda \in \widetilde{H}$.*

(*ii*) *There is a nonescaping-hyperbolic component* H_∞ *of* $\mathscr{F}_\infty$ *such that* $\widetilde{H} = H_\infty$.

Our result is a natural generalization of a theorem by Krauskopf and Kriete; they considered holomorphic families $\mathscr{F}_n = \{f_\lambda : \lambda \in \mathbb{C}\}$, $n \in (\mathbb{N} \cup \{\infty\})$, of entire maps for which the sets of singular values have the same finite cardinality and are holomorphically parametrized. They proved the same conclusions as in Theorem 7.1, provided that $\mathscr{F}_n \to \mathscr{F}_\infty$ uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$ [KK97]. Our proof of Theorem 7.1 follows the same idea as in [KK97].
The first case in Theorem 7.1 does indeed occur; an example is given in Section 7.3. Nevertheless, parameters that belong to a kernel define maps which exhibit certain stability. Under the same assumptions and notations as in Theorem 7.1, we prove the following result.

Theorem 7.2. *Let* λ *belong to a kernel* $\widetilde{H}$. *Then* $f_\lambda \in \mathscr{F}_\infty$ *is a J-stable function.*

Structure of Chapter 7

We start with preliminary concepts such as Hausdorff and kernel convergence in Section 7.1 and dynamical approximation in Section 7.2. Section 7.3 addresses the proofs of Theorems 7.1 and 7.2. In the final part, we discuss examples to which our results can be applied.
If not stated differently, we will assume that $f \in \mathrm{Hol}^*(\mathbb{C})$. Since we will consider nonescaping-hyperbolic maps, the concepts and results from Section 3.4 will be used frequently. Also, we would like to emphasize that for us, a complex manifold is in particular finite-dimensional.

7.1 Hausdorff and kernel convergence

Throughout this paragraph, let us assume that M is a metric space. The *Hausdorff distance* between two compact sets $A, B \subset M$ is defined by

$$d_H(A, B) := \inf\{\varepsilon > 0 : A \subset U_\varepsilon(B),\ B \subset U_\varepsilon(A)\}.$$

For studying convergence of open connected subsets of M we will define a concept analogous to the *Carathéodory* or *kernel convergence* for domains in the complex plane. For more details, see [Gol57, §5].

Definition 7.3 (Kernel). Let $o \in M$ and let $O_n \subset M$, $n \in \mathbb{N}$, be open connected sets containing o. The *kernel* of the sequence (O_n) (w.r.t. o) is the largest open connected set $O \ni o$ such that each compact set $K \subset O$ is contained in all but finitely many O_n.

We call the point o the *marked point* of the sequence (O_n). Clearly, the existence of a kernel is equivalent to the existence of a neighbourhood of o which is contained in all but finitely many O_n.
We say that the sequence (O_n) *converges* to O (*as kernels*) and write $O_n \to O$ if O is a kernel of each subsequence of (O_n). The middle example in Figure 7.1 gives an example of a sequence that does not converge to its kernel.
Observe that a sequence (O_n) can have more than one kernel (see the left-hand example in Figure 7.1), and each of them is specified by the choice of a marked point. Now let M be a locally compact metric space, e.g. an analytic manifold, and let O be a kernel of O_n. Since, by definition, every kernel is open and since M is locally compact, every point in O has a compact neighbourhood which is contained in O. This means that we can choose any point $o \in O$ to be the marked point. Hence we will talk about the sets O_n and O without mentioning the marked point if it is implicit from the context which point is meant.
We have the following relation between Hausdorff and kernel convergence.

Proposition 7.4. *Let K_n, K be nonempty compact subsets of a locally compact metric space M. Then $d_H(K_n, K) \to 0$ as $n \to \infty$ if and only if the following two conditions hold:*

- *every component O of $K^c := M \backslash K$ is a kernel of a sequence of components O_n of $K_n^c := M \backslash K_n$,*
- *every kernel of an infinite sequence (O_{n_k}) of components of $K_{n_k}^c$ is a component of K^c.*

We will omit the proof since it is elementary and follows mainly from the definitions of kernel and Hausdorff convergence, and since we do not require the statement for any proof in the thesis; its purpose is more the illustration of how the given concepts of convergence relate to each other. The right-hand example in Figure 7.1 shows the necessity of the second requirement in Proposition 7.4.

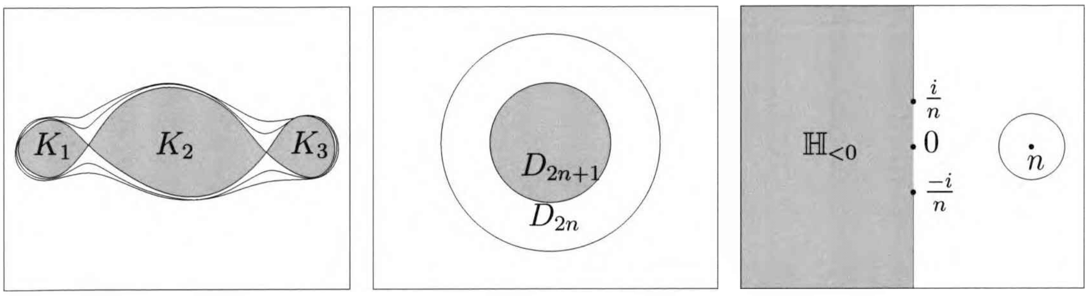

Figure 7.1: *left:* The interiors of the curves are simply-connected domains with three different kernels K_1, K_2 and K_3. *middle:* The domains $D_n := \{z : |z| < 2 - r_n \text{ where } n \equiv r_n \bmod 2, r_n \in \{0,1\}\}$ have a unique kernel D_1 but they do not converge to it. *right:* The domains $D_n := \mathbb{C} \setminus (\{z = iy : |y| \geq 1/n\} \cup \{z : |z-n| \leq 1\})$ converge as kernels to the left half-plane $\mathbb{H}_{<0}$ w.r.t. the marked point -1 but their complements do not converge to $\mathbb{C} \setminus \mathbb{H}_{<0}$ in the Hausdorff metric.

7.2 Dynamical approximation

We denote the *locally uniform distance* between $f, g \in \mathrm{Hol}(\mathbb{C})$ by $\chi_{\mathrm{luc}}(f, g)$. The metric $\chi_{\mathrm{luc}}(f, g)$ induces the topology of locally uniform convergence on $\mathrm{Hol}(\mathbb{C})$, so we say that f_n converge to f *locally uniformly* if and only if $\chi_{\mathrm{luc}}(f_n, f) \to 0$ as $n \to \infty$. (One can define this topology on a larger space like the set of all continuous selfmaps of $\mathbb{C}$ but we are only interested in entire maps. For details see e.g. [Mil06, Chapter 3].) It follows from the Weierstraß Approximation Theorem [Mil06, Theorem 1.4] that the space $\mathrm{Hol}(\mathbb{C})$ is closed with respect to this topology.

For entire maps with a non-empty set of singular values, we introduce a new metric which combines locally uniform convergence with controlled behaviour on the set of singular values. Hence this metric will be more convenient for the study of dynamics of entire functions.

Definition and Proposition 7.5. *The map* $\chi_{\mathrm{dyn}} : \mathrm{Hol}^*(\mathbb{C}) \to [0, \infty)$ *with*

$$\chi_{\mathrm{dyn}}(f, g) := \chi_{\mathrm{luc}}(f, g) + d_H(S(f), S(g))$$

is a metric, where $d_H(S(f), S(g))$ *is measured with respect to the spherical metric.*

We will say that the sequence f_n *approximates* f dynamically *if* $\chi_{\mathrm{dyn}}(f_n, f) \to 0$ *as* $n \to \infty$.

Maps which are close in the metric χ_{luc} do not necessarily have the property that their sets of singular values are close in the Hausdorff metric. This does not have to be true even in the case of a family of functions which depends holomorphically on some parameter λ, as the following example shows.

Example 7.6. Let $f_\lambda(z) = \mathrm{e}^{-\lambda z^2+z-2}$ with $\lambda \in \mathbb{C}$. The map $f_0(z) = \mathrm{e}^{z-2}$ has 0 as its only singular value. Now let $\lambda \neq 0$ be any complex number. Then apart from the asymptotic value 0, the map f_λ has the additional critical value $v_\lambda := \mathrm{e}^{1/(4\lambda)-2}$. Clearly, $v_\lambda \to +\infty$ when $\lambda \searrow 0$.

It is well-known that if f_n is a sequence of entire maps such that $\chi_{\text{luc}}(f_n, f) \to 0$, then for every $w \in S(f)$ there is a sequence $\{w_n : w_n \in S(f_n) \text{ for every } n\}$ such that $w_n \to w$ (see e.g. [Kis95, Theorem 2]), yielding lower semi-continuity for the sets of singular values in case of locally uniform convergence. Hence convergence in the metric χ_{dyn} makes in particular sure that there are no sequences of singular values of the approximating maps f_n that accumulate outside $S(f)$. Nonescaping-hyperbolicity is not an open property in the topology of locally uniform convergence. For instance, let $f_\lambda(z) := \mathrm{e}^{\lambda z} \cdot z^2$. Then $f_0(z) = z^2$ is nonescaping-hyperbolic while for any sufficiently small $\lambda > 0$, the critical value $4/(\mathrm{e}^2\,\lambda^2)$ escapes to ∞. However, the set of nonescaping-hyperbolic entire maps is open in the topology induced by the metric χ_{dyn}.

Theorem 7.7 (Nonescaping-hyperbolicity is an open property). *The set*

$$\mathcal{H} := \{f \in \mathrm{Hol}^*(\mathbb{C}) : f \textit{ is nonescaping-hyperbolic}\}$$

is open in the topology induced by the metric χ_{dyn}.

Note that $\mathcal{H} \subset \mathrm{Hol}^*_b(\mathbb{C})$. Theorem 7.7 will follow from the following lemma.

Lemma 7.8. *Let* $f \in \mathrm{Hol}^*(\mathbb{C})$ *and let* $K \subset \mathcal{A}(f)$ *be a compact set. Then* $K \subset \mathcal{A}(g)$ *for all* $g \in \mathrm{Hol}^*(\mathbb{C})$ *that are sufficiently close to* f *in the metric* χ_{luc}.

Proof. The components of $\mathcal{A}(f)$ form an open cover of the compact set K, hence there is a finite subcover. We can assume w.l.o.g. that K is contained in the basin of attraction $A(z_0)$ of a fixed point $z_0 \in \mathbb{C}$ of f, since otherwise we can repeat the argument for every attracting periodic point of f. There exists a bounded open set $U \ni z_0$ such that $f(U) \Subset U$. By definition of χ_{luc}, $g(U) \Subset U$ holds for all $g \in \mathrm{Hol}(\mathbb{C})$ for which $\chi_{\text{luc}}(f, g)$ is sufficiently small. By Montel's

Theorem, $\{g^k\}_{k\in\mathbb{N}}$ is normal in U and by the Contraction Mapping Theorem, g has a fixed point in U which is necessarily attracting. Since U is bounded, it follows that $U \subset \mathcal{A}(g)$.
There exists $N \in \mathbb{N}$ such that $f^N(K) \subset U$, so again, by locally uniform convergence, $g^N(K) \subset U \subset \mathcal{A}(g)$ if $\chi_{\text{luc}}(f,g)$ sufficiently small. The claim now follows from the complete invariance of $\mathcal{A}(g)$ under the map g. □

Proof of Theorem 7.7. Let f be nonescaping-hyperbolic, hence $S(f)$ is bounded and contained in $\mathcal{A}(f)$. Choose $\delta > 0$ sufficiently small such that $K := \overline{U_\delta(S(f))} \subset \mathcal{A}(f)$. Since K is compact, it follows from Lemma 7.8 that there is a constant $\varepsilon > 0$ such that $K \subset \mathcal{A}(g)$ for all g with $\chi_{\text{luc}}(f,g) < 2\varepsilon$. Now choose $\eta = \min\{\varepsilon, \delta\}$. Then for all maps g with $\chi_{\text{dyn}}(f,g) < \eta$ we obtain

$$S(g) \subset U_\delta(S(f)) \Subset \mathcal{A}(g),$$

hence g is nonescaping-hyperbolic. □

7.3 Stability of nonescaping-hyperbolic parameters

Recall that our goal is to prove that under certain conditions, a kernel of a sequence of nonescaping-hyperbolic components equals a nonescaping-hyperbolic component of the limit family. As we will see, it is not hard to show that every such component of $\mathscr{F}_\infty$ is contained in a kernel of a sequence of nonescaping-hyperbolic components of $\mathscr{F}_n$. This statement requires even less restrictions than those stated in Theorem 7.1. For the other inclusion to hold, we have to construct a more sensible setup.
Let us consider the set of sequences $\{a_0a_1\ldots : a_i \in \{0,1\}, a_i \leq a_{i+1}\}$. We can identify this set with $\mathbb{N}\cup\{\infty\}$: The number $n \in \mathbb{N}$ corresponds to the sequence defined by $a_i = 0$ for all $i \leq n$ and $a_{n+1} = 1$; the point at ∞ corresponds to the sequence $a_i \equiv 0$. We will denote this one-point compactification of $\mathbb{N}$ by $\widehat{\mathbb{N}}$. This space is a complete metric space with metric $\chi_{\widehat{\mathbb{N}}}(m,n) := 2^{-\min_k(m_k \neq n_k)}$.
From now on, we assume that M is a complex manifold with a metric χ_M and define $M' := \widehat{\mathbb{N}} \times M$. The relation $\chi_{M'}((m,\lambda),(n,\nu)) := \chi_{\widehat{\mathbb{N}}}(m,n) + \chi_{M'}(\lambda,\nu)$ then defines a metric $\chi_{M'}$ on M'.
For every $n \in \widehat{\mathbb{N}}$ let $\mathscr{F}_n = \{f_{n,\lambda} : \mathbb{C} \to \mathbb{C},\ \lambda \in M\} \subset \text{Hol}_b^*(\mathbb{C})$ be a family of functions parametrized by M. To simplify the notations, we will skip the first

index entry for maps in $\mathscr{F}_\infty$, i.e., we will write f_λ for $f_{\infty,\lambda} \in \mathscr{F}_\infty$. We want all families $\mathscr{F}_n, n \in \widehat{\mathbb{N}}$, to satisfy the following requirement.

Dynamical standing assumption ('dsa'). The map

$$F : M' \to \mathrm{Hol}_b^*(\mathbb{C}),\ (n, \lambda) \mapsto f_{n,\lambda}$$

is continuous with respect to the metrics $\chi_{M'}$ and χ_{dyn}.

The key-feature of 'dsa' is the local uniformity in λ *and* n: Let $f_{\lambda_0} \in \mathscr{F}_\infty$ and let $K \subset \mathbb{C}$ be a compact set. Then for every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $|f_{n,\lambda}(z) - f_{\lambda_0}(z)| < \varepsilon$ for all $z \in K$, $\lambda \in U_\delta(\lambda_0)$ and $n \geq n_0$.

Notations. For every $n \in \widehat{\mathbb{N}}$ we denote by

$$\mathcal{H}(\mathscr{F}_n) := \{\lambda \in M : f_{n,\lambda} \in \mathscr{F}_n \text{ is nonescaping-hyperbolic}\}$$

the parameters corresponding to nonescaping-hyperbolic maps in the respective family. We will usually denote a component of $\mathcal{H}(\mathscr{F}_n)$ by H_n, a component of $\mathcal{H}(\mathscr{F}_\infty)$ by H_∞ and a kernel of a sequence H_n by $\widetilde{H}$.

Proposition 7.9. *Let H_∞ be a component of $\mathcal{H}(\mathscr{F}_\infty)$. Then there exists a kernel $\widetilde{H}$ of a sequence of components of $\mathcal{H}(\mathscr{F}_n)$ such that $H_\infty \subset \widetilde{H}$.*

Proof. Let $\lambda_0 \in \mathcal{H}(\mathscr{F}_\infty)$. It follows from Theorem 7.7 and the dynamical standing assumption that there exists a neighbourhood $U(\lambda_0) \subset M$ such that $U(\lambda_0) \subset \mathcal{H}(\mathscr{F}_n)$ for all sufficiently large $n \in \mathbb{N}$. □

To prove the opposite inclusion, we have to make additional restrictions. Our requirements, that we will assume from now on, are formalized in the following way.

Holomorphic standing assumption ('hsa').

(*i*) For every $n \in \widehat{\mathbb{N}}$, the maps $f_{n,\lambda}$ depend holomorphically on $\lambda \in M$.

(*ii*) Let $n \in \mathbb{N}$ and $\lambda_0 \in M$. Then the singular values of f_{n,λ_0} are holomorphically parametrized by M, i.e., for each singular value s of f_{n,λ_0} there exists a holomorphic map $w_n : M \to \mathbb{C}, \lambda \mapsto w_n(\lambda)$ such that $w_n(\lambda_0) = s$ and $w_n(\lambda) \in S(f_{n,\lambda})$.

Note that the second condition of 'hsa' does not imply that a parametrization of $S(f_{n,\lambda_0})$ for some λ_0 needs to be an exhaustion of the set of singular values for another parameter $\lambda \neq \lambda_0$. A priori, it is possible that $S(f_{n,\lambda_0}) = \{w_n^i(\lambda_0)\}_{i\in I}$ for some index set I but $\{w_n^i(\lambda)\}_{i\in I} \subsetneq S(f_{n,\lambda})$ for some $\lambda \neq \lambda_0$.
Also note that we do not assume condition (ii) to be satisfied by the maps in $\mathscr{F}_\infty$. The reason is that we only need a *local* holomorphic parametrization of the sets of singular values for maps in $\mathscr{F}_\infty$ and as the next statement shows, this follows from the assumptions we already made.

Theorem 7.10. *Let $\widetilde{H}$ be a kernel of a sequence of nonescaping-hyperbolic components H_n, $U \ni \lambda_0$ a simply-connected neighbourhood of λ_0 with $U \Subset \widetilde{H}$, and let s be a singular value of f_{λ_0}.*
Then there exists a holomorphic map $w : U \to \mathbb{C}$ such that $w(\lambda) \in S(f_\lambda)$, $w(\lambda_0) = s$ and the family $\{f_\lambda^n(w(\lambda))\}_{n\in\mathbb{N}}$ is normal in U.

Proof. Let $f_{\lambda_0} \in \mathscr{F}_\infty$. By Theorem 2.5 and 2.6, the map f_{λ_0} has infinitely many repelling periodic points (in its Julia set). Let us pick two such points and denote them by $p(\lambda_0)$ and $q(\lambda_0)$; let n_1 and n_2 be their periods. If D is a disk at $p(\lambda_0)$ such that $p(\lambda_0)$ is the only periodic point of f_{λ_0} of period $\leq n_1$ in $\overline{D}$, then it follows from 'dsa' and Rouché's theorem that there is a neighbourhood $U(\lambda_0)$ of λ_0 and an integer $n_0 \geq 0$ such that for every $\lambda \in U(\lambda_0)$ and every $n \geq n_0$, the map $f_{n,\lambda}$ has exactly one periodic point $p_n(\lambda)$ of period n_1 in D and no other periodic point of period $\leq n_1$ in $\overline{D}$. By the Cauchy Integral Formula (after decreasing the initial disk D, if necessary), every such periodic point $p_n(\lambda)$ must be repelling. By the Implicit Function Theorem, every of these points can be analytically continued as a repelling periodic point of period n_1 in a sufficiently small neighbourhood. The previous observation then implies that for every sufficiently large $n \in \widehat{\mathbb{N}}$, there exists an analytic function $p_n : U(\lambda_0) \to \mathbb{C}$ such that $p_n(\lambda) \in D$ is a repelling periodic point of $f_{n,\lambda}$ of period n_1. By construction, $p_n(\lambda) \to p(\lambda)$ when $n \to \infty$. We can repeat the same procedure for $q(\lambda_0)$ and obtain holomorphic maps $q_n : U'(\lambda_0) \to \mathbb{C}$.
Let us now assume that $\lambda_0 \in \widetilde{H}$. Since $\widetilde{H}$ is open, there is a disk $B \subset \widetilde{H}$ centred at λ_0 such that the maps p_n, q_n are defined and holomorphic in B and their images are repelling periodic points of the corresponding maps (and periods). Let $U \supset B$ be a simply-connected bounded domain with closure in $\widetilde{H}$. Since $\widetilde{H}$ is a kernel of a sequence H_n of nonescaping-hyperbolic components,

the compact set $\overline{U}$ is contained in all H_n for $n \in \mathbb{N}$ chosen sufficiently large. Hence, the maps p_n and q_n ($n \in \mathbb{N}$) can be holomorphically continued to all of U, since otherwise the Implicit Function Theorem would imply that for some $\tilde{\lambda} \in U$, the map $f_{n,\tilde{\lambda}}$ has an indifferent periodic point.

Let us now consider the maps $\Phi_\lambda(z) = \frac{z-p(\lambda)}{q(\lambda)-p(\lambda)}$ and $\Phi_{n,\lambda}(z) = \frac{z-p_n(\lambda)}{q_n(\lambda)-p_n(\lambda)}$. Conjugating f_λ with Φ_λ and $f_{n,\lambda}$ with $\Phi_{n,\lambda}$, we obtain conformal conjugates such that the points 0 and 1 correspond to our previously considered repelling periodic points. Hence we can assume that $p(\lambda) = 0$, $q(\lambda) = 1$ for all $\lambda \in B$ and $p_n(\lambda) = 0$, $q_n(\lambda) = 1$ for all $\lambda \in U$. In particular,

$$\{0, 1\} \subset \mathcal{J}(f_\lambda), \mathcal{J}(f_{n,\lambda}) \text{ and } S(f_{n,\lambda}) \subset \mathbb{C}\backslash\{0, 1\}.$$

for all sufficiently large integers $n \in \widehat{\mathbb{N}}$ and the corresponding values of λ.

Let s be a singular value of f_{λ_0}. By 'dsa', there is a sequence of singular values s_n of the maps f_{n,λ_0} that converges to s. Due to 'hsa', there are holomorphic maps w_n such that $w_n(\lambda_0) = s_n$ and $w_n(\lambda) \in S(f_{n,\lambda})$ for all $\lambda \in M$. By the previous argument, we have that $w_n(U) \subset \mathbb{C}\backslash\{0, 1\}$, hence, by Montel's theorem, $\{w_n\}_{n\in\mathbb{N}}$ is a normal family on U. Let (w_{n_k}) be a convergent subsequence of (w_n), and let w be a limit function which is necessarily holomorphic. By construction we have that $w(\lambda_0) = s$, and 'dsa' implies that $w(\lambda) \in S(f_\lambda)$ holds for all $\lambda \in U$. Hence w is a holomorphic parametrization of the singular value s on U.

Consider now for a fixed ν the family $\{f^\nu_{n,\lambda}(w_n(\lambda))\}_{n\in\mathbb{N}}$ with $\lambda \in U$. Since the Fatou set of an entire map is completely invariant, $w_n(\lambda) \in \mathcal{F}(f_{n,\lambda})$ implies that $f^\nu_{n,\lambda}(w_n(\lambda)) \subset \mathcal{F}(f_{n,\lambda}) \subset \mathbb{C}\backslash\{0, 1\}$. Applying Montel's theorem it follows that for each ν the above family is normal in U. For simplicity, denote its limit by $S_\nu : U \to \widehat{\mathbb{C}}$, $\lambda \mapsto S_\nu(\lambda) := \lim_{n\to\infty} f^\nu_{n,\lambda}(w_n(\lambda))$. It follows from the local uniform convergence of the maps f_n and w_n that

$$S_\nu(\lambda) = \lim_{n\to\infty} \lim_{m\to\infty} f^\nu_{n,\lambda}(w_m(\lambda)) = \lim_{n\to\infty} f^\nu_{n,\lambda}(w(\lambda)) = f^\nu_\lambda(w(\lambda)).$$

By Hurwitz's theorem, either $S_\nu \subset \mathbb{C}\backslash\{0, 1\}$ or $S_\nu \equiv 0$ or 1. Recall that 0 and 1 are periodic points of f_λ, hence if $S_\nu \equiv 0$ (resp. 1) then there exists some $m \in \mathbb{N}$ such that $S_{\nu+km} \equiv 0$ (resp. 1) for all $k \in \mathbb{N}$. Applying Montel's theorem once more, we obtain that $\{S_\nu\}_{\nu\in\mathbb{N}}$ is a normal family on U. □

We can now prove the remaining statement which will then imply Theorem 7.1.

Theorem 7.11. *Let* $\widetilde{H}$ *be a kernel of a sequence of components of* $\mathcal{H}(\mathscr{F}_n)$*. If there is a component* H_∞ *of* $\mathcal{H}(\mathscr{F}_\infty)$ *such that* $\widetilde{H} \cap H_\infty \neq \emptyset$*, then* $\widetilde{H} \subset H_\infty$*.*

Proof. We will prove the statement by contradiction, so assume that there exists some $\lambda_0 \in \widetilde{H} \cap \partial H_\infty$. With the same notations as in the previous proof, it follows that $S_\nu(B) \subset \mathbb{C}\backslash\{0,1\}$ for all $\nu \in \mathbb{N}$, since $B' := B \cap H_\infty \neq \emptyset$ (and hence $S_\nu|_B$ cannot be constant 0 or 1).
For a limit function S of S_ν we have that either $S \equiv c \in \widehat{\mathbb{C}}$, in which case $S \equiv \infty$, or S is a non-constant function with $S(B) \subset \mathbb{C}\backslash\{0,1\}$.
The case $S \equiv \infty$ clearly cannot occur since this would mean that some singular value converges to ∞ for all parameters in B but the nonempty subset B' of B consists of nonescaping-hyperbolic parameters.
Let $S_\nu(B) \subset \mathbb{C}\backslash\{0,1\}$. For all $\lambda \in B'$ there is a holomorphic map $a : B' \to \mathbb{C}$ such that $a(\lambda)$ is an attracting point of f_λ which attracts $w(\lambda)$. Since $B \supset B'$ is simply-connected, the point $a(\lambda)$ can then be analytically continued to an attracting point on the entire domain B. The image $a(B)$ is bounded, hence for every $\lambda \in B$ the singular value $w(\lambda)$ is attracted by a finite attracting periodic cycle of f_λ. Since $s = w(\lambda_0)$ was assumed to be an arbitrary singular value of f_{λ_0}, we can repeat this procedure for any of the singular values of f_{λ_0}. Recall that, by assumption, $\mathscr{F}_\infty \subset \mathrm{Hol}_b^*(\mathbb{C})$, hence the singular sets are bounded. This implies that $B \ni \lambda_0$ is contained in some nonescaping-hyperbolic component H_∞ of $\mathscr{F}_\infty$, contradicting the assumption that $\lambda_0 \in \widetilde{H} \cap \partial H_\infty$.

□

Without the assumption of dynamical approximation (which is part of 'dsa') we cannot expect that a kernel $\widetilde{H}$ is always a nonescaping-hyperbolic component of the family $\mathscr{F}_\infty$. It is easy to find suitable examples. One such example was given in [KK95]: the authors approximated a holomorphic family of quadratic polynomials by families of polynomials of degree four, such that a kernel of a sequence of nonescaping-hyperbolic components was a proper subset of some nonescaping-hyperbolic component of the limit family.
Here we give an example which respects our standing assumptions, showing that the case $\widetilde{H} \cap \mathcal{H}(\mathscr{F}_\infty) = \emptyset$ in Theorem 7.1 does indeed occur.

Example 7.12. Let

$$P_{\lambda,n}(z) = z^3 - \left(\frac{\lambda-1}{\sqrt{\mu_n+\lambda-2}} + \sqrt{\mu_n+\lambda-2}\right) z^2 + \lambda z, \tag{7.1}$$

where $\lambda \in \mathbb{C} \setminus [1,5]$, $|\mu_n| < 1$ does not depend on λ and $\mu_n \to 1$ as $n \to \infty$ (for instance we can choose $\mu_n = \frac{n}{n+1}$). Note that $P_{\lambda,n}(z)$ and $P_\lambda(z) = z^3 - 2\sqrt{\lambda-1}z^2 + \lambda z$ satisfy 'dsa' and 'hsa'. Every $P_{\lambda,n}$ has 0 as a fixed point of multiplier λ. Furthermore, there is a second fixed point $a_n = \sqrt{\mu_n+\lambda-2}$ with multiplier μ_n. Thus, if we choose $\lambda \in \mathbb{D}$ then every polynomial $P_{\lambda,n}$ is nonescaping-hyperbolic. Hence there is a kernel $\widetilde{H}$ of components of $\mathcal{H}(\mathscr{F}_n)$ which contains the unit disk $\mathbb{D}$. On the other hand, every P_λ has a parabolic fixed point at $a = \sqrt{\lambda-1}$, hence $\mathcal{H}(\mathscr{F}_\infty) = \emptyset$.

Nevertheless, the behaviour of f_λ is still stable in the sense of *J-stability* for parameters belonging to a kernel. Here, J-stability is defined analogously to the case of rational maps or transcendental entire maps with finitely many singular values (see e.g. [EL92]):

Definition 7.13. Let $\mathscr{F} = \{f_\lambda : \mathbb{C} \to \mathbb{C} : \lambda \in M\}$ be a holomorphic family of entire functions. A map $f_{\lambda_0} \in G$ is said to be *J-stable* if $\mathcal{J}(f_{\lambda_0})$ moves holomorphically in a neighbourhood Λ of λ_0, i.e., if there is a holomorphic motion

$$\Phi : \Lambda \times \mathcal{J}(f_{\lambda_0}) \to \mathbb{C}$$

such that

$$\Phi(\lambda, \mathcal{J}(f_{\lambda_0})) = \mathcal{J}(f_\lambda) \text{ and } \Phi(\lambda, f_{\lambda_0}(z)) = f_\lambda(\Phi(\lambda, z))$$

for all $z \in \mathcal{J}(f_{\lambda_0})$ and all $\lambda \in \Lambda$.

Φ being a holomorphic motion means that Φ is injective in z when λ is fixed, holomorphic in λ for fixed z and that $\Phi_{\lambda_0} \equiv \text{id}$. For more details see [EL92, Section 8].
The map Φ_λ is a conjugacy between f_{λ_0} and f_λ on their Julia sets, hence it maps periodic points of f_{λ_0} to periodic points of f_λ. Since periodic points form a dense subset of the Julia set, such a conjugacy is unique if it exists.

Theorem 7.14. *Let $\widetilde{H}$ be a kernel of a sequence of components of $\mathcal{H}(\mathscr{F}_n)$ and let $\lambda \in \widetilde{H}$. Then $f_\lambda \in \mathscr{F}_\infty$ is J-stable.*

Proof. We will prove the contraposition, so let λ_0 be a parameter for which f_{λ_0} is not J-stable and let Λ be a simply-connected bounded neighbourhood of λ_0. Then there is some repelling periodic point, say of period n, of f_{λ_0} which has no analytic continuation as a solution of the equation $f_\lambda^n(z) - z = 0$. Otherwise, it would follow from the λ-lemma [EL92] that the closure of all repelling points of f_{λ_0}, which equals the Julia set of f_{λ_0}, would move holomorphically, contradicting that f_{λ_0} is not J-stable.
Let $p(\lambda_0)$ be such a repelling periodic point and let $\gamma : [0,1] \to \Lambda$, $\gamma(0) = \lambda_0$, be a path along which $p(\lambda_0)$ cannot be continued analytically. By the Implicit Mapping Theorem, the point $p(\lambda)$ for $\lambda = \gamma(1)$ must be indifferent. By the Minimum Modulus Principle we can then find a nearby path $\tilde{\gamma} \subset \Lambda$ connecting λ_0 to some parameter $\tilde{\lambda}$ along which the considered point becomes attracting. Hence there is a singular value $s(\tilde{\lambda})$ of $f_{\tilde{\lambda}}$ converging to the attracting periodic point $p(\tilde{\lambda})$.
Now let us assume that our assumption is wrong, meaning that λ_0 belongs to some kernel $\widetilde{H}$. We can assume w.l.o.g. that the neighbourhood Λ was chosen sufficiently small such that $\Lambda \Subset \widetilde{H}$. Then Theorem 7.10 implies that there is a holomorphic parametrization w of the singular value $s(\tilde{\lambda})$ such that $\{f_\lambda^n(w(\lambda))\}$ is a normal family on Λ. But then each $w(\lambda)$ converges to an attracting point $p(\lambda)$ of f_λ, in which case $p(\lambda)$ can be continued analytically to an attracting point of f_λ on the whole of Λ, contradicting the fact that $p(\lambda_0)$ is repelling. □

7.4 Construction of examples

As we have seen in the previous section, a sequence of families to which our results apply has to satisfy two primary conditions which we have formulated as the standing assumptions 'dsa' and 'hsa'. Hence starting with an entire map $f \in \mathrm{Hol}_b^*(\mathbb{C})$, we can split the problem into the following two:

(*i*) Find a sequence f_n of entire functions which approximates f *dynamically.*

(*ii*) Construct *holomorphic* families $\{f_{n,\lambda}\}$ and $\{f_\lambda\}$ using the functions f_n, f.

We will start with the second problem. It turns out that there is a natural way to find suitable holomorphic families for *any* entire function.

7.4.1 Families inside quasiconformal equivalence classes

Recall that M is a complex manifold. Let μ be a k-Beltrami coefficient of $\mathbb{C}$, i.e., $\mu : \mathbb{C} \to \mathbb{C}$ is a measurable function such that $\|\mu\|_\infty \leq k < 1$ almost everywhere (a.e.) in $\mathbb{C}$. By the Integrability Theorem [Leh86, Theorem 4.4], there exists a k-quasiconformal homeomorphism $\Psi : \mathbb{C} \to \mathbb{C}$ whose complex dilatation equals μ a.e. in $\mathbb{C}$.

Let $f \in \mathrm{Hol}_b^*(\mathbb{C})$ be an entire map. Then the *pull-back* $f^*\mu$ of μ by f is given by

$$f^*\mu(z) := (\mu \circ f)\frac{\overline{f'(z)}}{f'(z)}.$$

In particular, $\|\mu\|_\infty \leq k$ implies that $\|f^*\mu\|_\infty \leq k$.

By the Integrability Theorem there exists a quasiconformal homeomorphism Φ whose complex dilatation equals $f^*\mu$ a.e. in $\mathbb{C}$. A formal computation yields that the complex dilatation of the map $g = \Psi \circ f \circ \Phi^{-1}$ is 0 a.e., which means that g is a holomorphic map [Leh86, Theorem 1.1]. We say that g is *quasiconformally equivalent* to f.

Let $\Lambda \subset M$ be an open connected set and let $\{\Psi_\lambda\}$ be a family of quasiconformal homeomorphisms with uniformly bounded complex dilatations that depend holomorphically on $\lambda \in \Lambda$. The parametrized version of the Integrability Theorem [Hub06, Chapter 4.7] gives the following way to construct such a family, starting from a Beltrami coefficient μ_0:

Let $(\mu_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of Beltrami coefficients with $\|\mu_\lambda\|_\infty \leq k < 1$ that contains μ_0, i.e., $\mu_0 \equiv \mu_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. (One way to embed μ_0 in a holomorphic family of Beltrami coefficients is as follows: Let $h : \Lambda \to D_r(0)$ be a holomorphic map, where r is chosen such that $(1+r)\cdot\|\mu_0\|_\infty < 1$ and $h(\lambda_0) = 0$. Then the functions $\mu_\lambda := (1 + h(\lambda)) \cdot \mu_0$ form a holomorphic family of Beltrami coefficients.) For every λ, let Ψ_λ be a quasiconformal homeomorphism with Beltrami coefficient μ_λ, chosen such that all Ψ_λ have the same parametrization (e.g., all Ψ_λ fix 0, 1 and ∞). Then the Integrability Theorem implies that $\lambda \mapsto \Psi_\lambda$ is also holomorphic.

As in the previous construction we obtain a family $\mathscr{F}_\infty = \{f_\lambda = \Psi_\lambda \circ f \circ \Phi_\lambda^{-1}\}_{\lambda \in \Lambda}$ of entire maps. By computing the derivative of the equation $\Psi_\lambda \circ f = f_\lambda \circ \Phi_\lambda$ with respect to $\overline{\lambda}$, we get that the functions f_λ also depend holomorphically on λ (see e.g. the proof of Proposition 13 in [BC04]). Since an entire function f is a covering map when restricted to $\mathbb{C}\backslash f^{-1}(S(f))$, we obtain that $S(f_\lambda) = \Psi_\lambda(S(f))$, hence the set of singular values moves holomorphically in the family $\mathscr{F}_\infty$ in a unique way. Furthermore, $S(f_\lambda)$ is bounded for all λ since the maximal dilatations of the quasiconformal maps Ψ_λ are uniformly bounded.

Let (f_n) be a sequence of entire functions which converges to f dynamically. In the same manner as above, we define for each n the holomorphic family $\mathscr{F}_n = \{f_{n,\lambda} = \Psi_\lambda \circ f_n \circ \Phi_{n,\lambda}^{-1}\}_{\lambda\in\Lambda}$ by starting with the same family $\{\Psi_\lambda\}$ of quasiconformal homeomorphisms. As before, the set of singular values $S(f_{n,\lambda}) = \Psi_\lambda(S(f_n))$ moves holomorphically. Hence it remains to show that the map

$$F : M' \to \mathrm{Hol}_b^*(\mathbb{C}), \ (n, \lambda) \mapsto f_{n,\lambda}$$

is continuous. In other words, let $\lambda_0 \in M$ and let $(n,\lambda) \to (\infty, \lambda_0)$. We have to show that in this case,

$$\chi_{\mathrm{luc}}(f_{n,\lambda}, f_{\lambda_0}) + d_H(S(f_{n,\lambda}), S(f_{\lambda_0})) \to 0.$$

Clearly, $d_H(S(f_{n,\lambda}), S(f_{\lambda_0})) = d_H(\Psi_\lambda(S(f_n)), \Psi_{\lambda_0}(S(f))) \to 0$ as $(n,\lambda) \to (\infty, \lambda_0)$, since Ψ_λ depends holomorphically (so in particular continuously) on λ and the maps f_n approximate f dynamically.

To see that $\chi_{\mathrm{luc}}(f_{n,\lambda}, f_{\lambda_0}) \to 0$, we have to look at the sequence of pull-backs

$$f_n^*\mu_\lambda = (\mu_\lambda \circ f_n)\frac{\overline{f_n'(z)}}{f_n'(z)},$$

where μ_λ denotes the complex dilatation of Ψ_λ. By assumption, there exists a constant $k < 1$ such that $\|f_n^*\mu_\lambda\|_\infty < k < 1$ for all n. Since also $f_n^*\mu_\lambda \to f^*\mu_{\lambda_0}$ a.e., it follows that the (uniquely normalized solutions) $\Phi_{n,\lambda}$ converge locally uniformly to Φ_{λ_0} [Leh86, Theorem 4.6], yielding the desired statement.

7.4.2 Functions of sine type

The above construction yields many examples of families where Theorems 7.10 and 7.11 can be applied to, provided that we have a sequence of functions approximating f dynamically. There are clearly various ways of approximating an entire function locally uniformly but it is a very strong requirement to keep control over the sets of singular values, since this set can be arbitrarily complicated (for instance, it can have nonempty interior).

For certain (families of) transcendental entire functions that were and are of particular interest in holomorphic dynamics, appropriate approximations are known and were extensively studied. For instance, one can approximate the exponential map by the polynomials $P_n(z) = (1 + z/n)^n$ or the function $f(z) = \sin(z)/z$, which has infinitely many singular values, by the sequence $T_n(z)/z$, where $T_n(z)$ denotes the Chebyshev polynomial of degree n.

We want to introduce another set of transcendental entire functions for which a dynamical approximation exists, namely *real sine-type maps with real zeros.*

Definition 7.15. A function f is said to be of *sine type* σ if there are positive constants c, C, τ such that

$$c\,\mathrm{e}^{\sigma|\operatorname{Im} z|} \leq |f(z)| \leq C\,\mathrm{e}^{\sigma|\operatorname{Im} z|},$$

where the upper estimate holds everywhere in $\mathbb{C}$ and the lower estimate holds at least outside the horizontal strip $|\operatorname{Im} z| \leq \tau$.

Let f be a sine-type function and denote by z_n the zeros of f. By [Lev96, Lecture 17] or ([Sem07, Theorem 3]), the limit

$$\lim_{R\to\infty} \prod_{|z_n|\leq R} \left(1 - \frac{z}{z_n}\right)$$

exists uniformly on compact subsets of $\mathbb{C}$, and it defines an entire function called the *generating function of the sequence* (z_n) which equals f up to $K \cdot z^m$ where K is a constant and $m \geq 0$ is an integer [Sem07, Theorem 2]. By definition, the zeros of f are contained in a horizontal strip around the real axis and f has exactly two tracts over ∞, each of which contains some upper and lower half-plane, respectively. It follows from the Ahlfors-Denjoy Theorem [Nev53, XI, §4, 269] that f has at most two asymptotic values. The derivative of the generating function $\tilde{f}$ of (z_n) is given by $\tilde{f}' = \tilde{f} \cdot \sum \frac{1}{z-z_n}$ and an elementary computation shows that $\tilde{f}$ and hence f has no critical points outside a sufficiently wide horizontal strip. So the set of critical values of f is bounded, implying that every f belongs to the Eremenko-Lyubich class $\mathscr{B}$.

It is now easy to show the following.

Proposition 7.16. *Let f be a real sine-type function for which all zeros are real. Then there exists a sequence p_n of polynomials such that $\chi_{\mathrm{dyn}}(p_n, f) \to 0$ when $n \to \infty$.*

Proof. By a theorem of Laguerre [Tit32, Chapter 8.51], all zeros of $f'(z)$ are real and are separated from each other by the zeros of f. By basic calculus arguments, the generating polynomials cannot have "free" critical values, and by [Kis95, Theorem 2] each singular value of f is approximated by a sequence of singular values of the generating polynomials. Hence the sets of singular values converge in the Hausdorff metric. □

7.5 Questions and remarks

Recall that nonescaping-hyperbolicity is not an open property in the topology of locally uniform convergence. This example also shows that — in the same topology — the set of all *hyperbolic* maps in $\mathrm{Hol}^*(\mathbb{C})$ is not open either. Now let p be a hyperbolic polynomial for which a finite singular value escapes to infinity. It is plausible that for any sufficiently small ε, the neighbourhood $U_\varepsilon(p) = \{f \in \mathrm{Hol}^*(\mathbb{C}) : \chi_{\mathrm{dyn}}(f, p) < \varepsilon\}$ of p contains transcendental entire functions with the same property, i.e., maps for which some singular value escapes.

Conjecture 7.17. *The set $\{f \in \mathrm{Hol}^*(\mathbb{C}) : f$ is hyperbolic$\}$ is not open in the topology induced by χ_{dyn}.*

From here on, let $\mathscr{F}_n$ be families that satisfy 'dsa' and 'hsa', and let $\widetilde{H}$ denote a kernel of a sequence of components of $\mathcal{H}(\mathscr{F}_n)$. Recall from Example 7.12 that — by turning attracting fixed points into persistent parabolic fixed points of the limit family — we constructed a kernel with the property that none of the corresponding maps in the limit family is nonescaping-hyperbolic. The question is whether this is the only way to construct case (i) in Theorem 7.1.

Question 7.18. *Suppose that $\widetilde{H} \cap \mathcal{H}(\mathscr{F}_\infty) = \emptyset$. Is there a persistent parabolic periodic point in the family $\mathscr{F}_\infty$?*

It is also worth noticing that the mechanism that makes Example 7.12 work does not occur for families constructed in Section 7.4.1 using quasiconformal maps.

Question 7.19. *Suppose that the considered families can be written as* $\mathscr{F}_n = \{f_\lambda = \Psi_\lambda \circ f \circ \Phi_{n,\lambda}^{-1}\}$*, where* $\Psi_\lambda, \Phi_{n,\lambda}^{-1}$ *are quasiconformal homeomorphisms. Does* $\widetilde{H} \cap \mathcal{H}(\mathscr{F}_\infty) \neq \emptyset$ *hold?*

Let f be an arbitrary entire map. As we have seen in Section 7.4, it is easy to construct examples to which our results apply, provided there exists a sequence of maps that dynamically approximates f. However, none of the classical approximation methods in function theory makes sufficient statement about the relation of the sets of singular values. This leaves open an interesting question.

Question 7.20. *Is there a generic set of (transcendental) entire functions for which a nontrivial dynamical approximation exists?*

List of symbols

$\mathbb{N}$	natural numbers (including 0)
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers (complex plane)
$\widehat{\mathbb{C}}$	Riemann sphere $\mathbb{C} \cup \{\infty\}$
$\mathbb{C}^*$	punctured plane $\mathbb{C}\backslash\{0\}$
$\mathbb{D}$	unit disk $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$
$\mathbb{D}^*$	punctured disk $\mathbb{D} \setminus \{0\}$
$D_r(c)$	Euclidean disk centred at $c \in \mathbb{C}$ with radius r
S^1	unit circle $\partial\mathbb{D}$
$\mathbb{H}$	upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$
∂A	boundary of a set $A \subset \mathbb{C}$ relative to $\mathbb{C}$
$\overline{A}$	closure of a set $A \subset \mathbb{C}$ relative to $\mathbb{C}$
$\widehat{\partial} A$	boundary of a set $A \subset \widehat{\mathbb{C}}$ relative to $\widehat{\mathbb{C}}$
$\widehat{A}$	closure of a set $A \subset \widehat{\mathbb{C}}$ relative to $\widehat{\mathbb{C}}$
A'	derived set of a set $A \subset \mathbb{C}$, defined as the set of all finite limit points of A
$U_\varepsilon(A)$	ε-neighbourhood of a subset A of a metric space M
$\operatorname{dist}(A, B)$	Euclidean distance between two sets A and B
$A \Subset B$	$\overline{A}$ is a compact subset of the open set B
$\rho_U(z)$	density of the hyperbolic metric in the domain U

$\ell_U(\gamma)$	hyperbolic length of a rectifiable curve $\gamma \subset U$
$d_U(z,w)$	hyperbolic distance between two points $z, w \in U$
$\|Df(z)\|_U$	derivative of a holomorphic map f w.r.t. the hyperbolic metric on U
$\deg(f,z)$	local degree of f at the point z
$\rho(f)$	order of growth of f
$A(f)$	set of asymptotic values of f
$C(f)$	set of critical values of f
$S(f)$	set of singular values of f
$P(f)$	postsingular set of f
$\mathrm{Hol}(\mathbb{C})$	space of all entire functions
$\mathrm{Hol}^*(\mathbb{C})$	space of all nonconstant, nonlinear entire functions
$\mathrm{Hol}_b^*(\mathbb{C})$	space of all nonconstant, nonlinear entire functions with bounded singular sets
$\mathscr{B}$	Eremenko-Lyubich class
R^3S	class of all finite compositions of finite order maps in the class $\mathscr{B}$
$\mathcal{F}(f)$	Fatou set of f
$\mathcal{J}(f)$	Julia set of f
$I(f)$	Escaping set of f
$S_\mathcal{F}$	$S(f) \cap \mathcal{F}(f)$
$S_\mathcal{J}$	$S(f) \cap \mathcal{J}(f)$
$P_\mathcal{F}$	$P(f) \cap \mathcal{F}(f)$
$P_\mathcal{J}$	$P(f) \cap \mathcal{J}(f)$

$O^+(A)$	forward orbit of a set $A \subset \mathbb{C}$
$\mu(z)$	multiplier of z
$\mathrm{Attr}(f)$	set of attracting periodic points of f
$\mathrm{Par}(f)$	set of parabolic periodic points of f
$A(Z)$	basin of attraction of the attracting cycle Z
$A^*(Z)$	immediate attracting basin of the attracting cycle Z
$\mathcal{A}(f)$	set of points that converge to a cycle in $\mathrm{Attr}(f)$
$\mathcal{P}(f)$	set of points that converge nontrivially to a cycle in $\mathrm{Par}(f)$
$\mathscr{S}$, $\mathscr{S}(f, D, \alpha)$	static partition (of f w.r.t. D and α)
$\mathrm{addr}(z)$, $\mathrm{addr}_{\mathscr{S}}(z)$	external address of z (w.r.t. $\mathscr{S}$)
$\mathscr{D}$, $\mathscr{D}(f, D)$	dynamical partition (of f w.r.t. D)
$\mathrm{itin}_{\mathscr{D}}(z)$, $\mathrm{itin}(z)$	itinerary of z (w.r.t. $\mathscr{D}$)
σ	one-sided shift map
$\mathrm{lcm}\{n_i\}$	least common multiple of a sequence $(n_i) \subset \mathbb{N}$

8 References

[AO93] J. M. Aarts and L. G. Oversteegen, *The geometry of Julia sets*, Trans. Amer. Math. Soc. **338** (1993), no. 2, 897–918.

[Bak68] I. N. Baker, *Repulsive fixpoints of entire functions*, Math. Z. **104** (1968), 252–256.

[Bar07] K. Barański, *Trees and hairs for some hyperbolic entire maps of finite order*, Math. Z. **257** (2007), 33–59.

[BC04] X. Buff and A. Chéritat, *Upper bound for the size of quadratic Siegel disks*, Inventiones Mathematicae **156** (2004), no. 1, 1–24.

[BDH+00] C. Bodelón, R. L. Devaney, M. Hayes, G. Roberts, L. R. Goldberg, and J. H. Hubbard, *Dynamical convergence of polynomials to the exponential*, Journal of Difference Equations and Applications **6** (2000), 275–307.

[Ber93] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. **29** (1993), no. 2, 151–188, arXiv:math.DS/9310226.

[Ber02] ———, *A question of Eremenko and Lyubich concerning completely invariant domains and indirect singularities*, Proc. Amer. Math. Soc. **130** (2002), 3231–3236.

[BHK+93] W. Bergweiler, M. Haruta, H. Kriete, H. G. Meier, and N. Terglane, *On the limit functions of iterates in wandering domains*, Ann. Acad. Sci. Fenn., Ser A I Math. **18** (1993), 369–375.

[BKL91] I. N. Baker, J. Kotus, and Y. Lü, *Iterates of meromorphic functions iii: Preperiodic domains*, Ergodic Theory Dynamical Systems **11** (1991), 603–618.

[BP78] A. F. Beardon and C. Pommerenke, *The Poincaré metric of plane domains*, J. London Math. Soc. (2) **18** (1978), no. 3, 475–483.

[CL04] E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, 1. paperback ed., Cambridge University Press, 2004.

[DGH86] R. L. Devaney, L. R. Goldberg, and J. H. Hubbard, *A dynamical approximation to the exponential map by polynomials*, Preprint, Berkley (1986).

[DH84] A. Douady and J. H. Hubbard, *Étude dynamique des polynômes complexes I & II*, Pub. Math. d'Orsay (1984).

[DO08] D. Drasin and Y. Okuyama, *Singularities of Schröder maps and unhyperbolicity of rational functions*, Comput. Methods Funct. Theory **8** (2008), no. 1, 285–302, arXiv:math.DS/0704.3309.

[Dou93] A. Douady, *Descriptions of compact sets in C*, Topological methods in modern mathematics, Proceedings of a symposium in honor of John Milnor's sixtieth birthday (L. Goldberg and A. Phillips, eds.), Publish or Perish, Houston, Texas, 1993, pp. 429–465.

[DT86] R. L. Devaney and F. Tangerman, *Dynamics of entire functions near the essential singularity*, Ergodic Theory Dynamical Systems **6** (1986), 489–503.

[EL90] A. E. Eremenko and M. Yu. Lyubich, *Dynamics of analytic transformations*, Leningrad Math. J. **3** (1990), no. 1, 563–633.

[EL92] ———, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier Grenoble **42** (1992), no. 4, 989–1020.

[EM88] C. J. Earle and C. McMullen, *Quasiconformal isotopies*, Holomorphic Functions and Moduli (1988), 143–154.

[Eps93] A. L. Epstein, *Towers of Finite Type Complex Analytic Maps*, Ph.D. thesis, City University of New York, 1993.

[Eps97] ———, *Dynamics of finite type complex analytic maps I: Global structure theory*, Preprint, 1997.

[Ere89] A. E. Eremenko, *On the iteration of entire functions*, Dynamical Systems and Ergodic Theory, Proc. Banach Center Publ. Warsaw **23** (1989), 339–345.

[Fat19] P. Fatou, *Sur les équations fonctionelles*, Bull. Soc. Math. France **47** (1919), 161–271, **48** (1920), 33-94, 208-314.

[GK86] L. R. Goldberg and L. Keen, *A finiteness theorem for a dynamical class of entire functions*, Ergodic Theory Dynamical Systems **6** (1986), 183–192.

[GO70] A. A. Goldberg and I. V. Ostrovskii, *Value distribution of meromorphic functions*, (English translation), originally appeared in Nauka, Moscow, 1970.

[Gol57] G. M. Golusin, *Geometrische Funktionentheorie*, VEB Deutscher Verlag Der Wissenschaften, 1957.

[GS98] J. Graczyk and G. Świątek, *The real Fatou conjecture*, no. 144, Annals of Mathematics Studies, Princeton University Press, 1998.

[Hub06] J. H. Hubbard, *Teichmüller theory and applications to geometry, topology and dynamics, Volume I: Teichmüller theory*, Matrix Editions, Ithaca, New York, 2006.

[Jor82] T. Jorgensen, *Simple geodesics on Riemann surfaces*, Proc. Amer. Math. Soc. **86** (1982), no. 1, 120–122.

[Kis95] M. Kisaka, *Local uniform convergence and convergence of Julia sets*, Nonlinearity **8** (1995), 273–281.

[KK95] B. Krauskopf and H. Kriete, *On the convergence of hyperbolic components in families of finite type*, Mathematica Gottingensis - Schriftenreihe des Mathematischen Instituts der Universität Göttingen (1995).

[KK97] ______, *Kernel convergence of hyperbolic components*, Ergodic Theory Dynamical Systems **17** (1997), no. 6, 1137–1146.

[KS08] M. Kisaka and M. Shishikura, *On multiply connected wandering domains of entire functions*, Transcendental dynamics and complex analysis (P. J. Rippon and G. M. Stallard, eds.), no. 348, London Mathematical Society Lecture Notes Series, Cambridge University Press, 2008.

[Leh86] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Graduate Texts in Mathematics 109, Springer Verlag - New York, Berlin, Heidelberg, 1986.

[Lev96] B. Ya. Levin, *Lectures on entire functions*, Translations of Mathematical Monographs, American Mathematical Society, 1996.

[LZ98] J. K. Langley and J.H. Zheng, *On the fixpoints, multipliers and value distribution of certain classes of meromorphic functions*, Ann. Acad. Sci. Fenn. **23** (1998), 135–150.

[McM94] C. T. McMullen, *Complex dynamics and renormalization*, no. 135, Annals of Mathematics Studies, Princeton University Press, 1994.

[McM00] ______, *Hausdorff dimension and conformal dynamics II: Geometrically finite rational maps*, Comm. Math. Helv. **75** (2000), no. 1, 535–593.

[Mil06] J. Milnor, *Dynamics in one complex variable*, 3. ed., no. 160, Annals of Mathematics Studies, Princeton University Press, 2006.

[MSS83] R. Mañé, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Annales scientifiques de l'École Normale Supérieure, Sér. 4 (1983).

[Nev53] R. Nevanlinna, *Eindeutige analytische Funktionen*, 2. ed., Springer-Verlag Berlin, Göttingen, Heidelberg, 1953.

[PM97] R. Pérez-Marco, *Fixed points and circle maps*, Acta. Math. **179** (1997), no. 2, 243–294.

[Poi90] H. Poincaré, *Sur une classe nouvelle de transcendantes uniformes*, J. Math. Pures Appliquées IV Ser. **6** (1890), 316–365.

[Rem] L. Rempe, *Rigidity of escaping dynamics for transcendental entire functions*, to appear in Acta Math., arXiv:math.DS/0605058.

[Rem03] ———, *Dynamics of Exponential Maps*, Ph.D. thesis, Christian-Albrechts-Universität zu Kiel, 2003.

[Rem06] ———, *Topological dynamics of exponential maps on their escaping sets*, Ergodic Theory Dynamical Systems **26** (2006), no. 6, 1939–1975, arXiv:math.DS/0309107.

[Rem07a] ———, *On a question of Eremenko concerning escaping components of entire functions*, Bull. London Math. Soc. **39** (2007), 661–666, arXiv:math.DS/0610453.

[Rem07b] ———, *On nonlanding dynamic rays of exponential maps*, Ann. Acad. Sci. Fenn. **32** (2007), arXiv:math.DS/0511588.

[Rem08] ———, *Siegel disks and periodic rays of entire functions*, Journal für Reine und Angewandte Mathematik **624** (2008), 81–102, arXiv:math.DS/0408041.

[RRRS] G. Rottenfusser, J. Rückert, L. Rempe, and D. Schleicher, *Dynamic rays of bounded type entire functions*, to appear in Ann. Math., arXiv:math.DS/0704.3213.

[RS08] G. Rottenfußer and D. Schleicher, *Escaping points of the cosine family*, Transcendental dynamics and complex analysis (P. J. Rippon and G. M. Stallard, eds.), no. 348, London Mathematical Society Lecture Notes Series, Cambridge University Press, 2008.

[Sch97] W. Schwick, *Repelling periodic points in the Julia set*, Bull. London Math. Soc. **29** (1997), no. 3, 314–316.

[Sch07a] D. Schleicher, *The dynamical fine structure of iterated cosine maps and a dimension paradox*, Duke Math. Journal **136** (2007), no. 2, 343–356, arXiv:math.DS/0406255.

[Sch07b] ———, *Hausdorff dimension, its properties, and its surprises*, American Mathematical Monthly **114** (2007), 509–528, arXiv:math.DS/0505099.

[Sem07] G. Semmler, *Complete interpolating sequences, the discrete Muckenhoupt condition, and conformal mapping*, Preprint (2007), arXiv:math.DS/0708.2787.

[SZ03] D. Schleicher and J. Zimmer, *Periodic points and dynamic rays of exponential maps*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 327–354.

[Thu79] W. P. Thurston, *Geometry and topology of three-manifolds*, Princeton Lecture Notes (1979).

[Thu84] ———, *On the combinatorics and dynamics of iterated rational maps*, Preprint (1984).

[Tit32] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, 1932.

[Val54] G. Valiron, *Fonctions analytiques*, 1. ed., Presses Universitaires de France, 1954.

[Why71] G. T. Whyburn, *Analytic topology*, 1. ed., vol. XXVIII, American Mathematical Society Colloquium Publications, 1971.